"In mathematics you don't understand things…

…you just get used to them."

- John von Neumann

(http://en.wikipedia.org/wiki/John_von_Neumann)

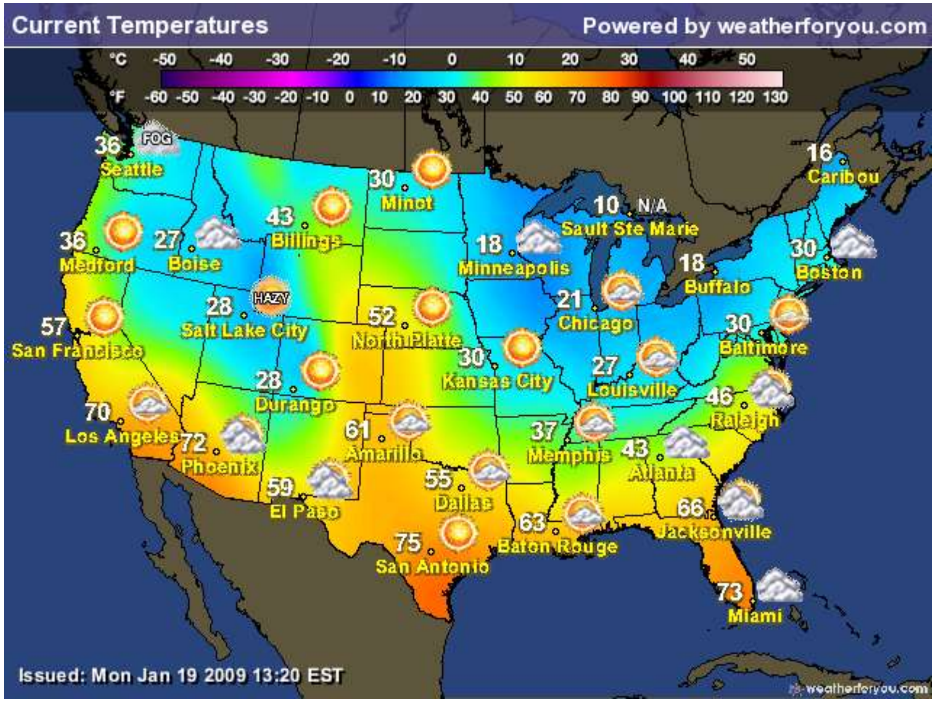
Vector Analysis

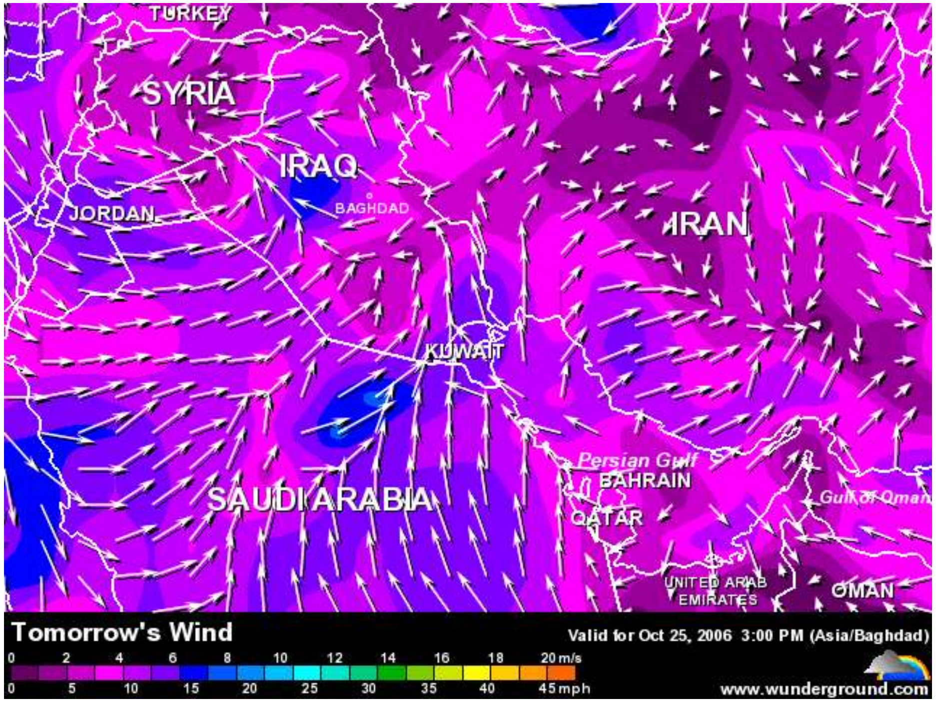
30 Jan.	**Vector Analysis**	3-1 thru 3-3	**Lab: Analyzer This**
1 Feb.	"	3-4, 3-5	
3 Feb.	"	3-6, 3-7	

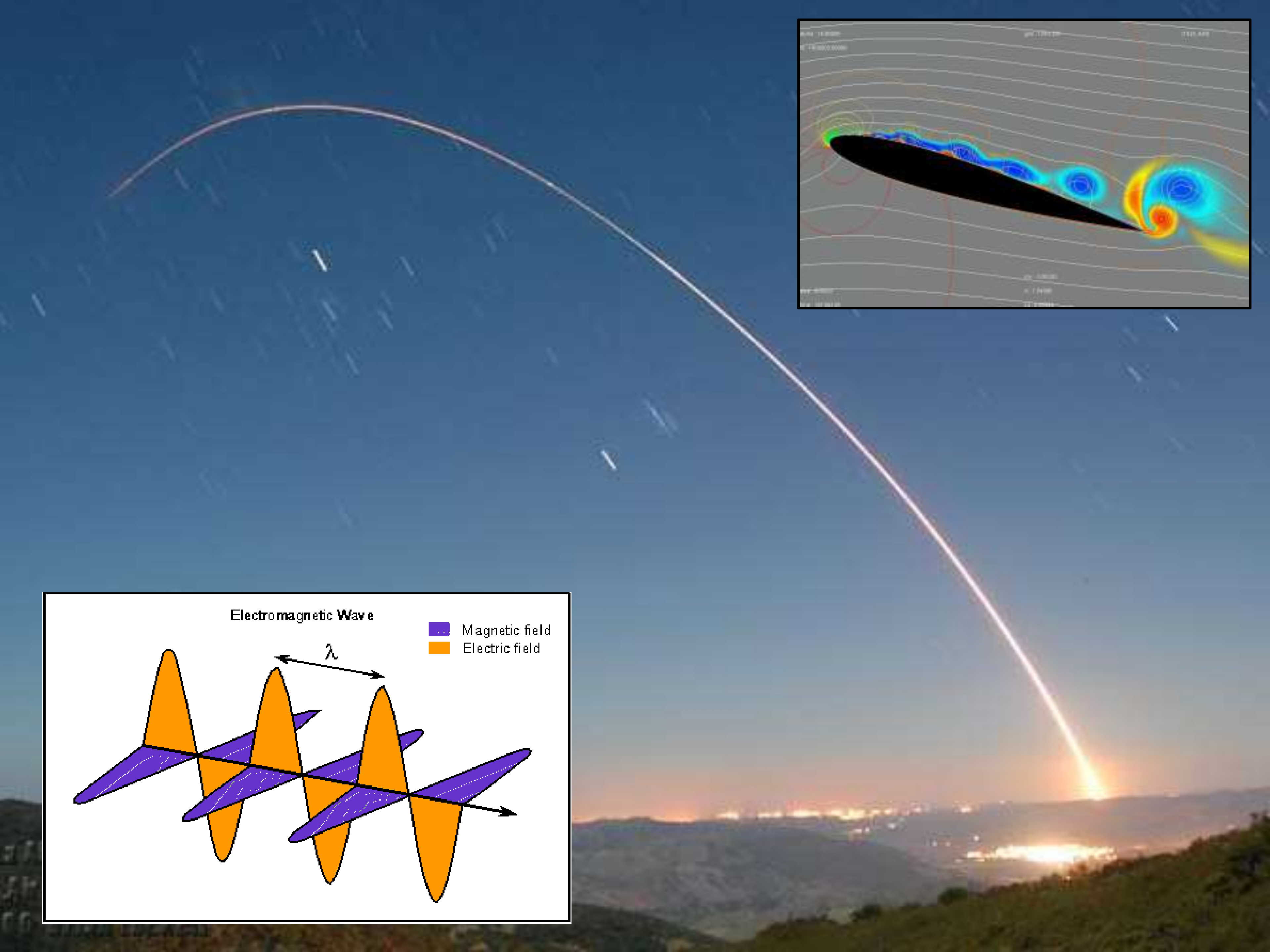
Scalar – a quantity whose value is represented by a single real number (temperature, mass, volume)

Vector – a quantity whose value is represented by a magnitude and a direction in space (force, velocity, position vector)

Field – a function of the coordinates of every point in a region in space that defines a scalar or a vector quantity (temperature in a room, Earth's gravitational field)






Electromagnetic Wave
Magnetic field
Electric field
λ

VECTORS

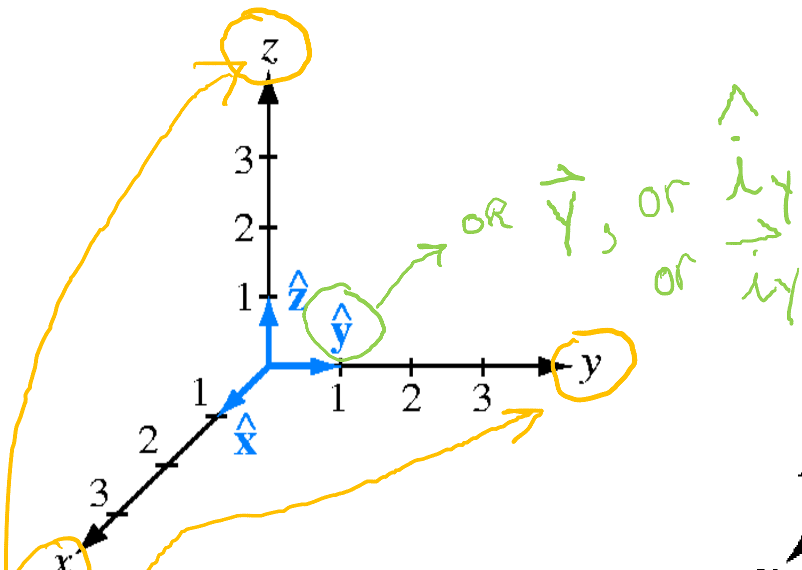
$$|A|^2 = A_x^2 + A_y^2 + A_z^2$$

$|A|$ = magnitude of $\vec{A}$

Note: a vector exists at a single point in space-time

$\Rightarrow \vec{A}(x, y, z, t)$

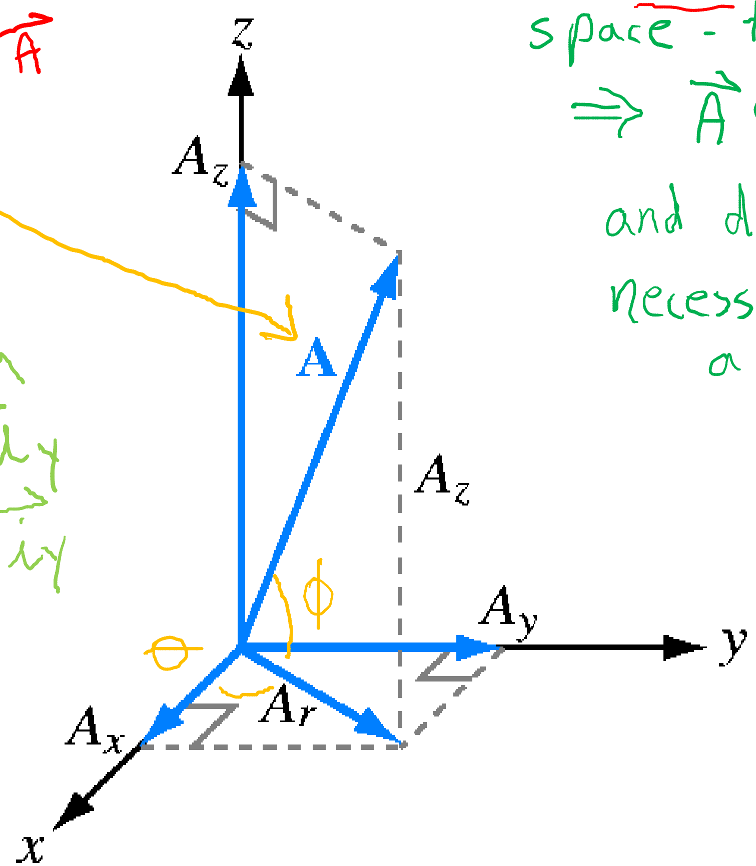
and does not necessarily point to a location!



(a) Base vectors

(or unit vectors)

Cartesian Coordinates



(b) Components of $\mathbf{A} = \vec{A}$

$= A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

vector

vector

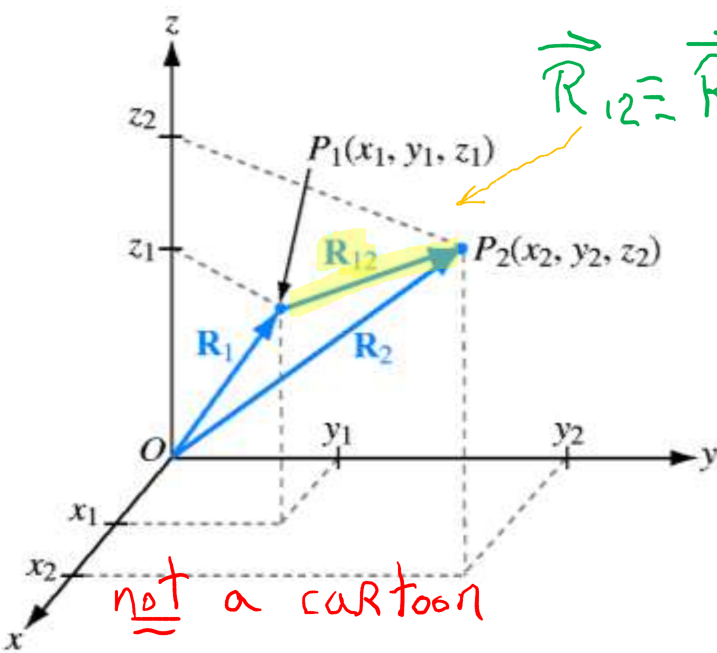
easier than bold when writing

Let $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$
$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$ } in Cartesian coordinates

Q: What does $\vec{A} = \vec{B}$ mean??

Special Case

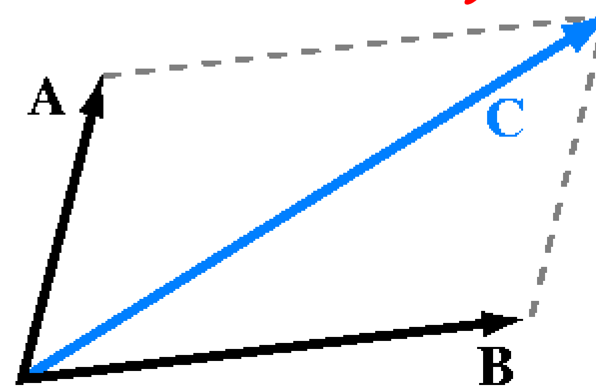
Position Vector, $\vec{R}$, is Vector from origin to point in space, P.



not a cartoon

$\vec{R}_{12} = \vec{R}_2 - \vec{R}_1$

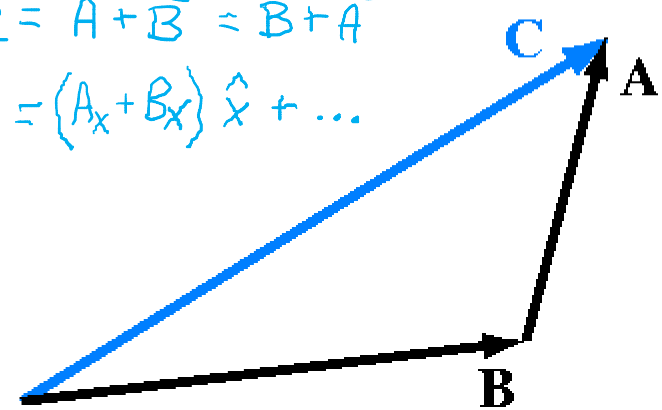
Note: this is a cartoon



(a) Parallelogram rule

$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$

$= (A_x + B_x)\hat{x} + \ldots$

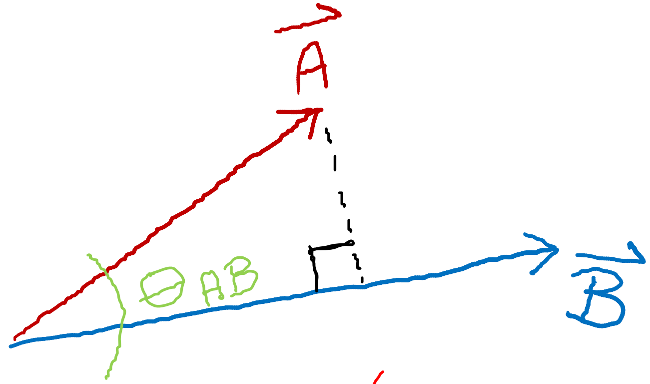


(b) Head-to-tail rule

$\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Dot Product

$$\underbrace{\vec{A}\cdot\vec{B}}_{\text{scalar}} = \vec{B}\cdot\vec{A}$$



projection of $\vec{A}$ onto $\vec{B}$

$$|\vec{A}||\vec{B}|\cos\theta_{AB}$$

Note: the dot product of any 2 normal* vectors is zero!

* perpendicular...

$$\vec{A}\cdot\vec{B} = \underbrace{A_xB_x + A_yB_y + A_zB_z}_{\text{for Cartesian Coords.}}$$

$$\vec{A}\cdot\vec{A} = |\vec{A}|^2$$

Why will we care about Dot products?

Example:

Find the angle between $\vec{A} = \hat{x}3 + \hat{y} - \hat{z}2$ and $\vec{B} = -\hat{x} + \hat{y} + \hat{z}5$

$|\vec{A}| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}$, $|\vec{B}| = \sqrt{(-1)^2+1^2+5^2} = \sqrt{27}$, $|\vec{A}||\vec{B}|\cos\phi = \sqrt{378}\cos\phi$

$\vec{A}\cdot\vec{B} = 3(-1)+1\cdot1-2\cdot5 = -3+1-10 = -12$

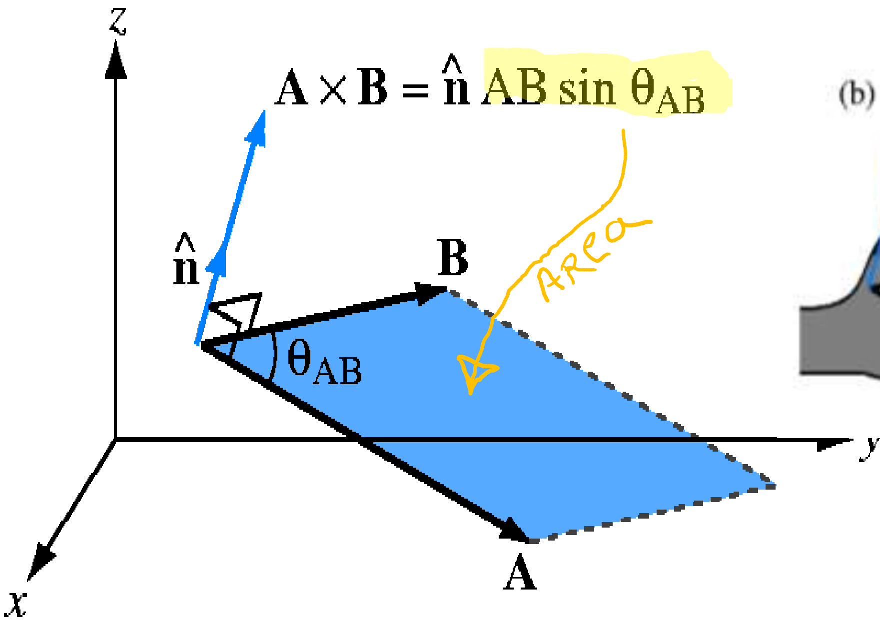
$\cos\phi = \frac{-12}{\sqrt{378}}$, $\phi \approx 128°$

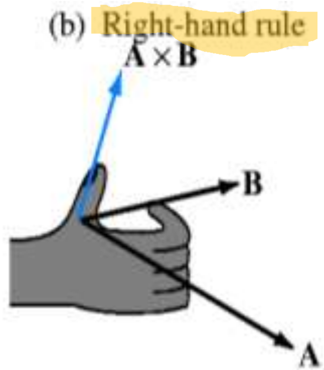
Cross-Product

$$\vec{A} \times \vec{B} = \hat{n}\,|\vec{A}||\vec{B}| \sin\theta_{AB} = -\vec{B}\times\vec{A}$$

$\underbrace{\vec{A}\times\vec{B}}_{\text{a vector}}$

$\hat{n}$ ← normal to both $\vec{A}$ & $\vec{B}$





$\vec{A} \times \vec{A} = ?$

$\hat{x} \times \hat{y} = ?$

$$\vec{A}\times\vec{B} = \underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}_{\text{for Cartesian}}$$

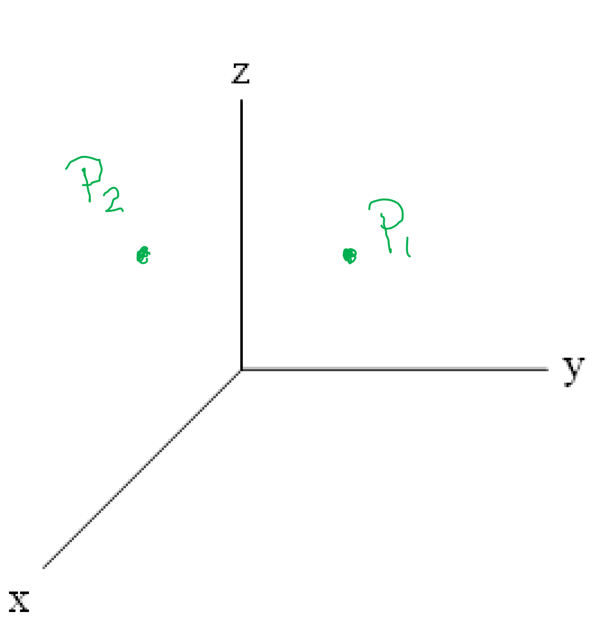
determinant

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \begin{matrix} \hat{x} & \hat{y} \\ A_x & A_y \\ B_x & B_y \end{matrix}$$

$$(A_yB_z - A_zB_y)\,\hat{x}$$
$$+ (A_zB_x - A_xB_z)\,\hat{y}$$
$$+ (A_xB_y - A_yB_x)\,\hat{z}$$

Why will we care about cross-products?

Exercise 3.1 Find the distance vector between $P_1(1,2,3)$ and $P_2(-1,-2,3)$ in Cartesian coordinates.



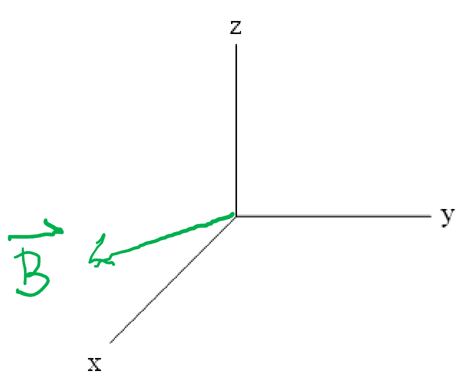
$$\vec{R}_{12} = \overrightarrow{P_1P_2} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)$$
$$= \hat{\mathbf{x}}(-1-1) + \hat{\mathbf{y}}(-2-2) + \hat{\mathbf{z}}(3-3)$$
$$= -\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4.$$

What is the distance between P_1 & P_2?

$$\Rightarrow |\vec{R}_{12}| = \sqrt{2^2+4^2} \cong 4.47 \text{ units}$$

Exercise 3.3 Find the angle that vector **B** of Example 3-1 makes with the z-axis.

$$\vec{B} = -\hat{x} - 5\hat{y} - \hat{z}$$

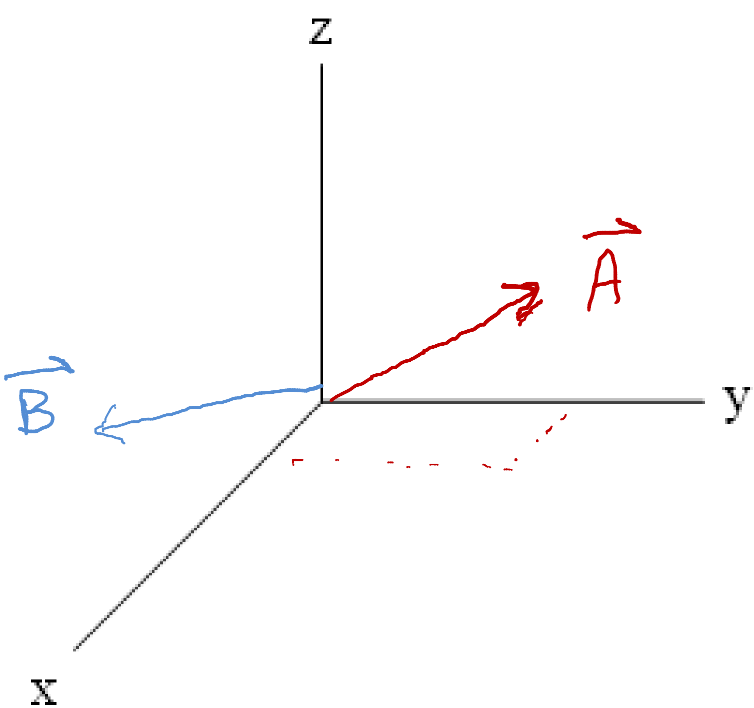


$$\mathbf{B} \cdot \hat{\mathbf{z}} = B\cos\theta$$
$$(-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = \sqrt{27}\cos\theta$$
$$\cos\theta = \frac{-1}{\sqrt{27}}$$
$$\theta = 101.1^\circ.$$

Exercise 3.2 Find the angle θ between vectors $\mathbf{A}$ and $\mathbf{B}$ of Example 3-1 using the cross product between them.

$\vec{A} = 2\hat{x} + 3\hat{y} + 3\hat{z}$

$\vec{B} = -\hat{x} - 5\hat{y} - \hat{z}$



$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$$

$$\sin\theta_{AB} = \frac{|\mathbf{A} \times \mathbf{B}|}{AB}$$
$$= \frac{|(\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3) \times (-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}})|}{\sqrt{22}\sqrt{27}}$$
$$= \frac{|-\hat{\mathbf{z}}10 + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}3 - \hat{\mathbf{x}}3 - \hat{\mathbf{y}}3 + \hat{\mathbf{x}}15|}{\sqrt{22}\sqrt{27}}$$
$$= \frac{|\hat{\mathbf{x}}12 - \hat{\mathbf{y}} - \hat{\mathbf{z}}7|}{\sqrt{22}\sqrt{27}} = \frac{\sqrt{144 + 1 + 49}}{\sqrt{22}\sqrt{27}} = 0.57$$

$$\theta_{AB} = \sin^{-1}(0.57) = 34.9^\circ \text{ or } 145.1^\circ.$$

Q: how would you find a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$?

Examples:

Find all unit vectors that are perpendicular to both

$$\vec{A} = \hat{x}3 + \hat{y} - \hat{z}2 \quad \text{and} \quad \vec{B} = -\hat{x} + \hat{y} + \hat{z}5$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 1 & -2 \\ -1 & 1 & 5 \end{vmatrix} = \hat{x}(5+2) + \hat{y}(2-15) + \hat{z}(3+1) = \hat{x}7 - \hat{y}13 + \hat{z}4$$

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{\left|\vec{A} \times \vec{B}\right|} = \frac{\hat{x}7 - \hat{y}13 + \hat{z}4}{\sqrt{7^2 + (-13)^2 + 4^2}} = \frac{\hat{x}7 - \hat{y}13 + \hat{z}4}{\sqrt{234}} = \hat{x}0.46 - \hat{y}0.85 + \hat{z}0.26$$

$$-\hat{n} = -\hat{x}0.46 + \hat{y}0.85 - \hat{z}0.26$$

$$\hat{n} \perp \left(\vec{A}, \vec{B}\right) \quad \text{and} \quad -\hat{n} \perp \left(\vec{A}, \vec{B}\right)$$

Which of the following products of vectors do not make sense:

a) $\left(\vec{A} \cdot \vec{B}\right) \times \vec{C}$

b) $\left(\vec{A} \times \vec{B}\right) \times \vec{C}$

c) $\vec{A}\left(\vec{B} \cdot \vec{C}\right)$

d) $\vec{A} / \vec{B}$

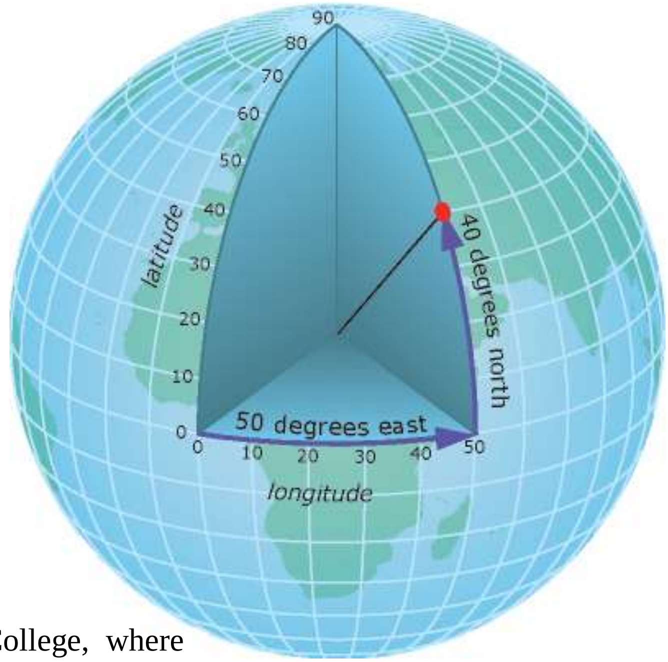
e) $\vec{A} / \hat{a}$

f) $\left(\vec{A} \times \vec{B}\right) \cdot \vec{C}$

a) A scalar cannot be crossed with a vector
d) and e) Division of vectors is not defined

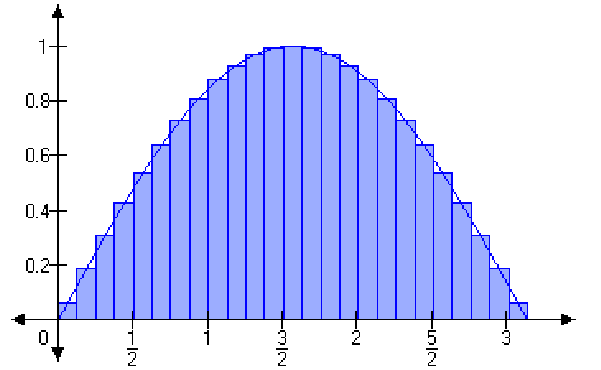
Orthogonal Coordinate Systems

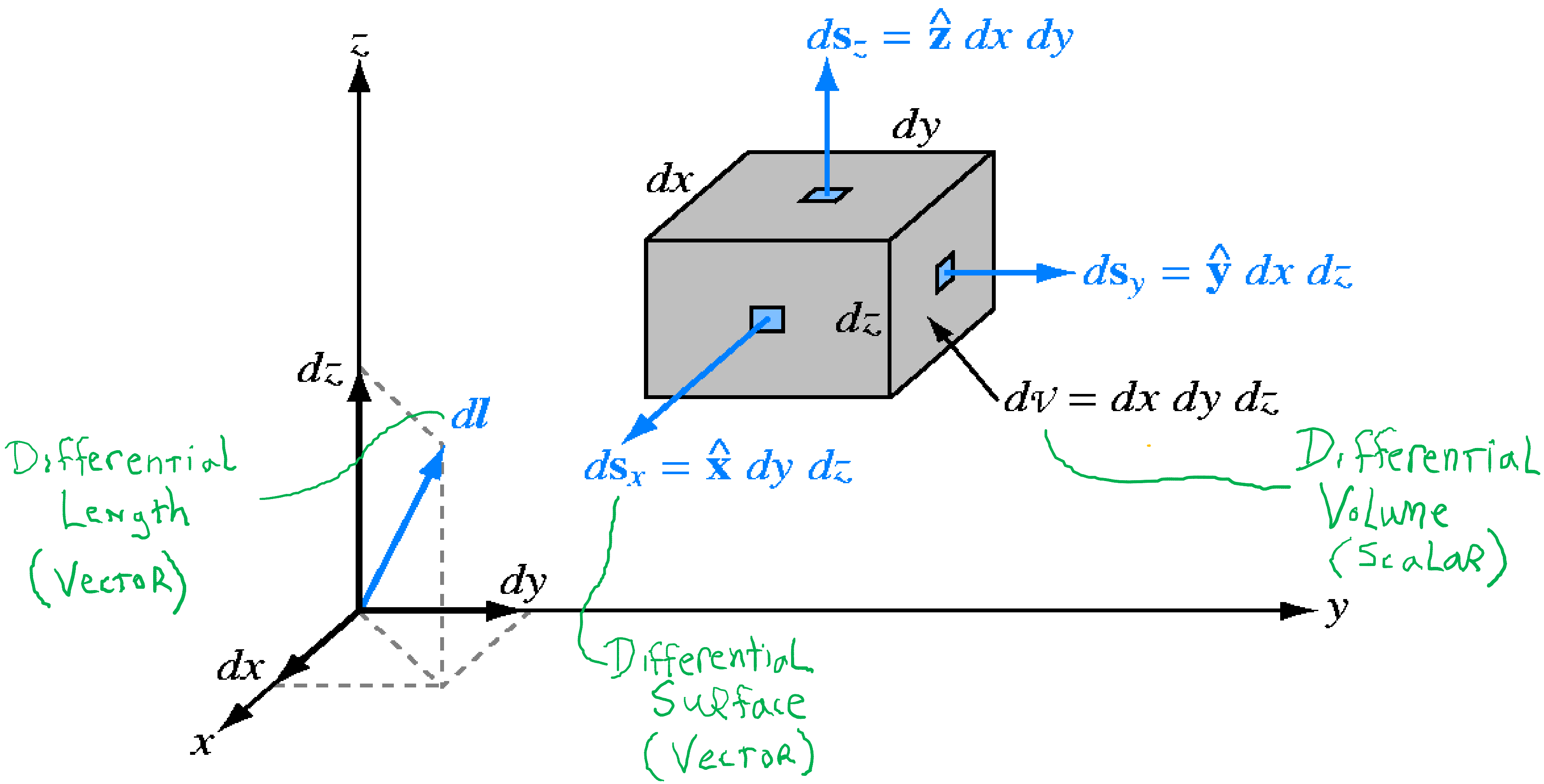
http://en.wikipedia.org/wiki/Ren%C3%A9_Descartes



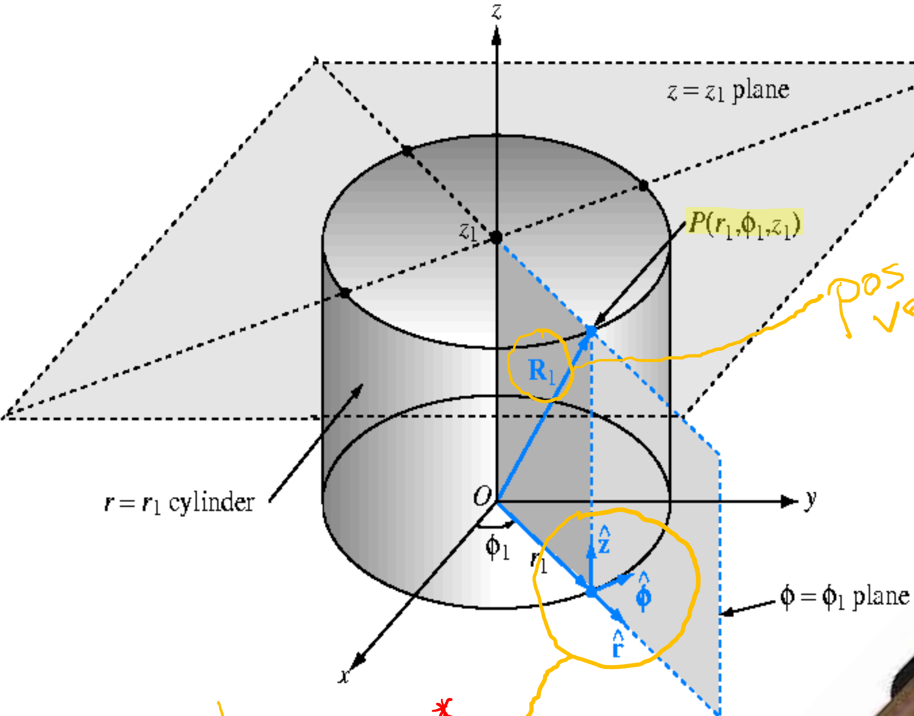
Q: if you went straight through the Earth from State College, where would you be on the other side? What would be the nearest big city?

Calculus


1
0.8
0.6
0.4
0.2
0
1/2
1
3/2
2
5/2
3


z
x
y
dx
dy
dz
dl
Differential Length (Vector)
dx
dy
dz
ds_z = ẑ dx dy
ds_y = ŷ dx dz
ds_x = x̂ dy dz
dv = dx dy dz
Differential Surface (Vector)
Differential Volume (Scalar)

* Note Difference

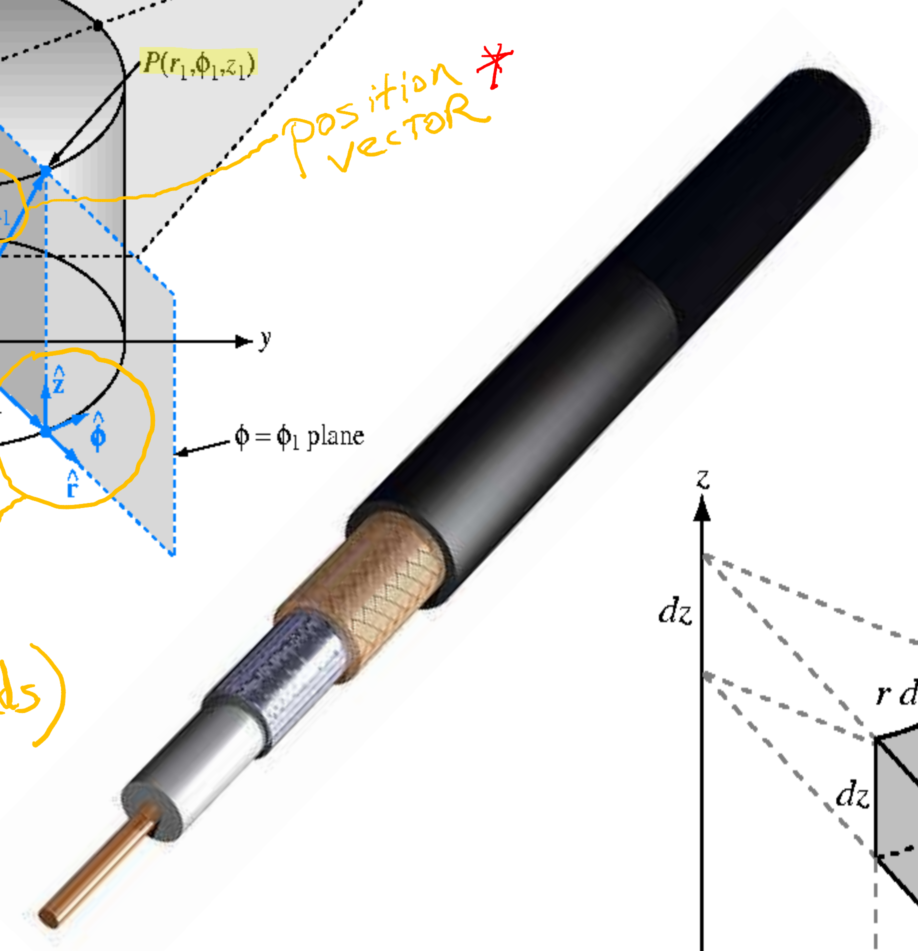


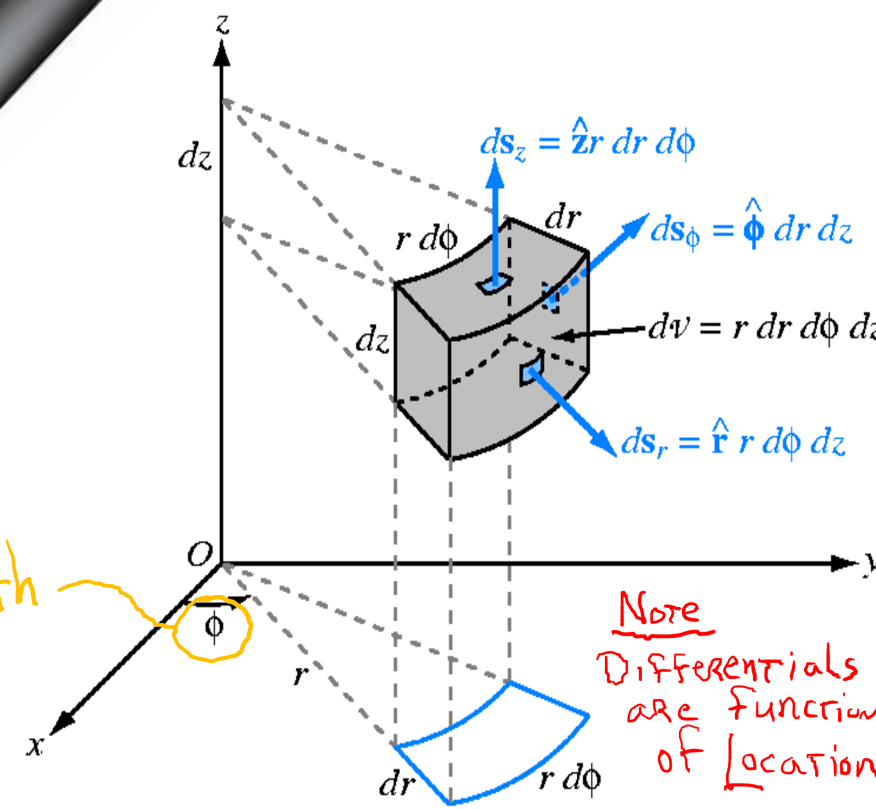
position vector *

Orthonormal Basis Set (Cylindrical Coords) *

- $\hat{r}$ & $\hat{\phi}$ are functions of ϕ !

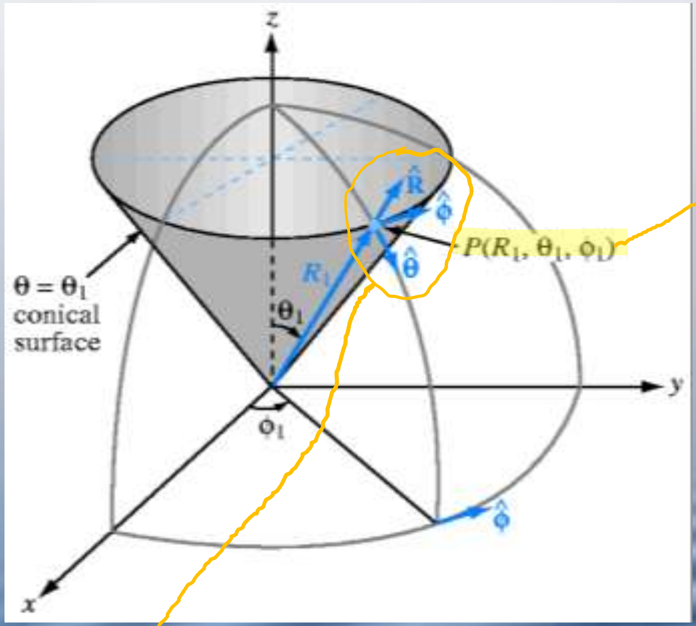
- what is $\hat{z} \times \hat{r}$, etc.?





Azimath

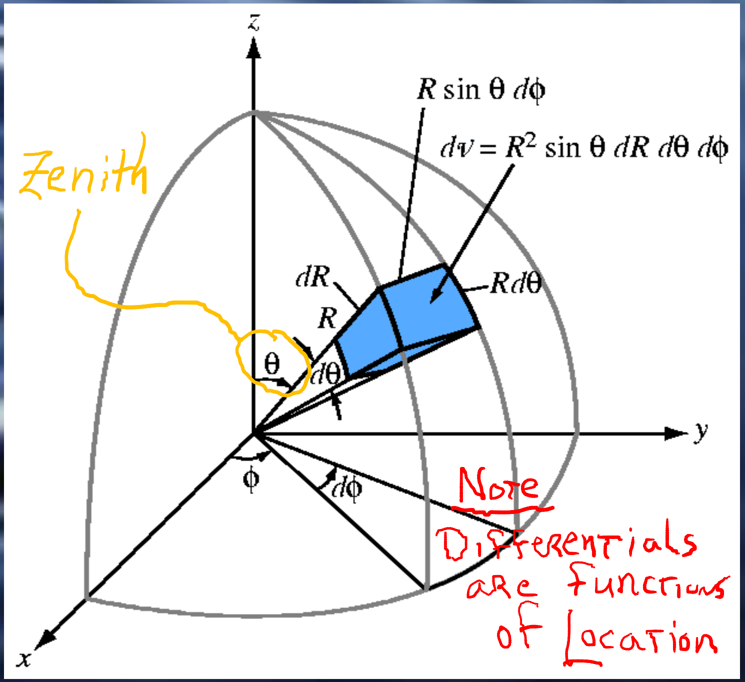
Note
Differentials are functions of Location



position

ortho basis (spherical)

- All 3 depend on ϕ and θ



	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A \rvert =$	$+\sqrt{A_x^2 + A_y^2 + A_z^2}$	$+\sqrt{A_r^2 + A_\phi^2 + A_z^2}$	$+\sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V} =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Problem 3.22 Use the appropriate expression for the differential surface area ds to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \leq \phi \leq \pi/3;\ -2 \leq z \leq 2,$

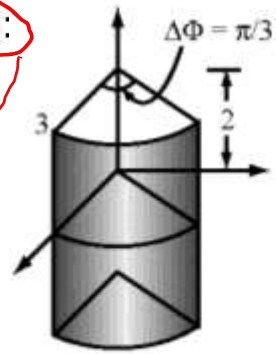
(b) $2 \leq r \leq 5;\ \pi/2 \leq \phi \leq \pi;\ z = 0,$

(c) $2 \leq r \leq 5;\ \phi = \pi/4;\ -2 \leq z \leq 2,$

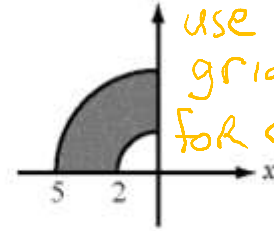
(d) $R = 2;\ 0 \leq \theta \leq \pi/3;\ 0 \leq \phi \leq \pi,$

(e) $0 \leq R \leq 5;\ \theta = \pi/3;\ 0 \leq \phi \leq 2\pi.$

Also sketch the outlines of each of the surfaces.

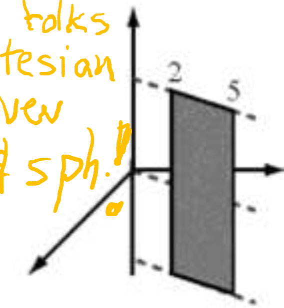


(a)

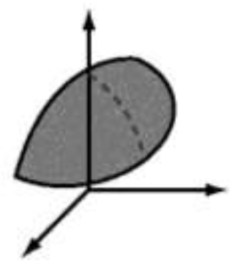


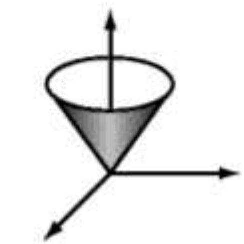
(b)

Note how folks use cartesian grids, even for cyl. & sph.!



(c)


(d)


(e)

(a) Using Eq. (3.43a),

$$A = \int_{z=-2}^{2}\int_{\phi=0}^{\pi/3} (r)|_{r=3}\, d\phi\, dz = \left((3\phi z)\Big|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq. (3.43c),

$$A = \int_{r=2}^{5}\int_{\phi=\pi/2}^{\pi} (r)|_{z=0}\, d\phi\, dr = \left(\left(\tfrac{1}{2} r^2 \phi\right)\Big|_{r=2}^{5} \right)\Big|_{\phi=\pi/2}^{\pi} = \frac{21\pi}{4}.$$

(c) Using Eq. (3.43b),

$$A = \int_{z=-2}^{2}\int_{r=2}^{5} (1)|_{\phi=\pi/4}\, dr\, dz = \left((rz)\Big|_{z=-2}^{2} \right)\Big|_{r=2}^{5} = 12.$$

(d) Using Eq. (3.50b),

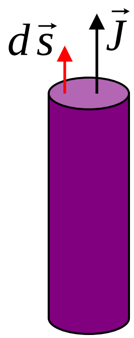
$$A = \int_{\theta=0}^{\pi/3}\int_{\phi=0}^{\pi} \left(R^2 \sin\theta\right)\Big|_{R=2}\, d\phi\, d\theta = \left((-4\phi\cos\theta)\Big|_{\theta=0}^{\pi/3} \right)\Big|_{\phi=0}^{\pi} = 2\pi.$$

(e) Using Eq. (3.50c),

$$A = \int_{R=0}^{5}\int_{\phi=0}^{2\pi} (R\sin\theta)|_{\theta=\pi/3}\, d\phi\, dR = \left(\left(\tfrac{1}{2} R^2 \phi \sin\frac{\pi}{3}\right)\Big|_{\phi=0}^{2\pi} \right)\Big|_{R=0}^{5} = \frac{25\sqrt{3}\pi}{2}.$$

Examples:

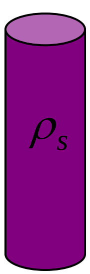
Find the total current flowing through a copper wire that has a circular cross-section with radius r_0 = 1 mm. The current density is J = 10 mA/mm^2.



$$I = \int_S \vec{J}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{r=0}^{r_0} J\, r\, dr\, d\phi = 2\pi J \int_{r=0}^{r_0} r\, dr = 2\pi J\left(\frac{r^2}{2}\right)_0^{r_0} = 2\pi J\,\frac{r_0^{\,2}}{2} =$$

$$= 2\pi \times 10^4 \times \frac{0.001^2}{2} = 31.4\,\mathrm{mA}$$

Find the total electric charge on the surface of a plastic rod if the surface charge density is ρ_s =2 pC/m^2. The radius of the rod is r_0 = 5 mm and its length is l = 10 cm.

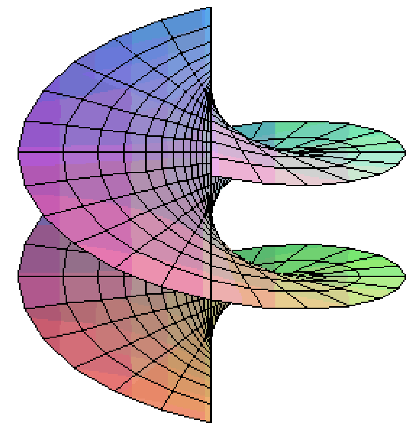


$$Q = \int_S \rho_s\, ds = \int_{S_{cylinder}} \rho_s\, ds + 2\times\left(\int_{S_{top}} \rho_s\, ds\right) = \int_{z=0}^{l}\int_{\phi=0}^{2\pi} \rho_s r_0\, d\phi\, dz + 2\times \int_{\phi=0}^{2\pi}\int_{r=0}^{r_0} \rho_s\, r\, dr\, d\phi =$$

$$= 2\pi\rho_s r_0 (z)_0^l + 2\times 2\pi\rho_s\left(\frac{r^2}{2}\right)_0^{r_0} = 2\pi\rho_s r_0 l + 4\pi\rho_s\frac{r_0^{\,2}}{2} = 2\pi\rho_s r_0(l + r_0)$$

$$= 2\pi\times 2\times 10^{-9}\times 0.005\times(0.1+0.005) = 6.6\,\mathrm{pC}$$

Problem 3.26 At a given point in space, vectors **A** and **B** are given in spherical coordinates by



$$\mathbf{A} = \hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}},$$
$$\mathbf{B} = -\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3.$$

Find:

(a) the scalar component, or projection, of **B** in the direction of **A**,

(b) the vector component of **B** in the direction of **A**,

(c) the vector component of **B** perpendicular to **A**.

(a) Scalar component of **B** in direction of **A**:

$$C = \mathbf{B}\cdot\hat{\mathbf{a}} = \mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|} = (-\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3)\cdot\frac{(\hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}})}{\sqrt{16+4+1}}$$
$$= \frac{-8-3}{\sqrt{21}} = -\frac{11}{\sqrt{21}} = -2.4.$$

(b) Vector component of **B** in direction of **A**:

$$\mathbf{C} = \hat{\mathbf{a}}C = \mathbf{A}\frac{C}{|\mathbf{A}|} = (\hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}})\,\frac{(-2.4)}{\sqrt{21}}$$
$$= -(\hat{\mathbf{R}}2.09 + \hat{\boldsymbol{\theta}}1.05 - \hat{\boldsymbol{\phi}}0.52).$$

(c) Vector component of **B** perpendicular to **A**:

$$\mathbf{D} = \mathbf{B} - \mathbf{C} = (-\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3) + (\hat{\mathbf{R}}2.09 + \hat{\boldsymbol{\theta}}1.05 - \hat{\boldsymbol{\phi}}0.52)$$
$$= \hat{\mathbf{R}}0.09 + \hat{\boldsymbol{\theta}}1.05 + \hat{\boldsymbol{\phi}}2.48.$$

Problem 3.18 Use arrows to sketch each of the following vector fields:

(a) $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$,

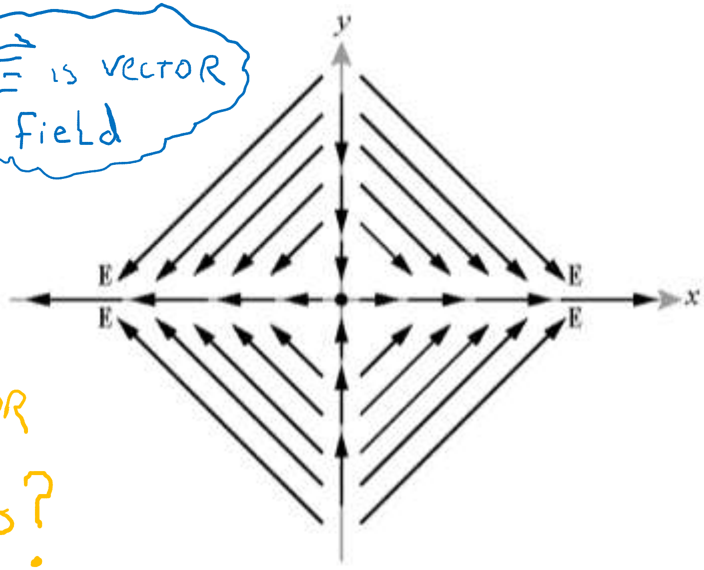
(b) $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$,

(c) $\mathbf{E}_3 = \hat{\mathbf{y}}\frac{1}{x}$,

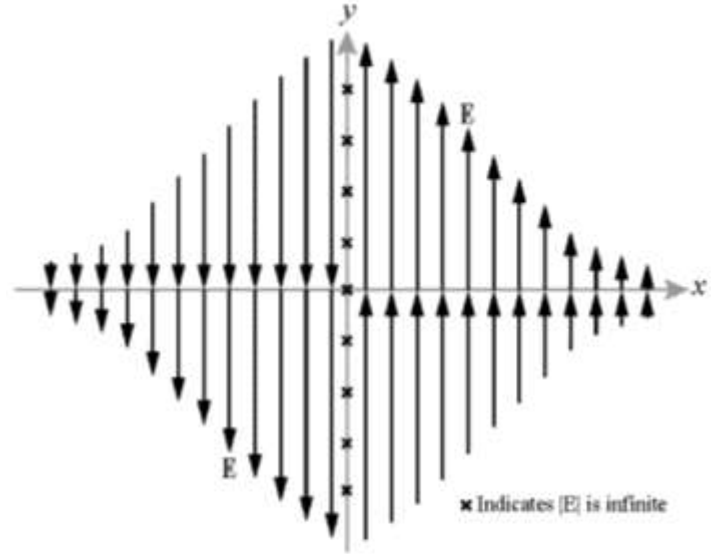
(d) $\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$.

$\vec{E}$ is vector field

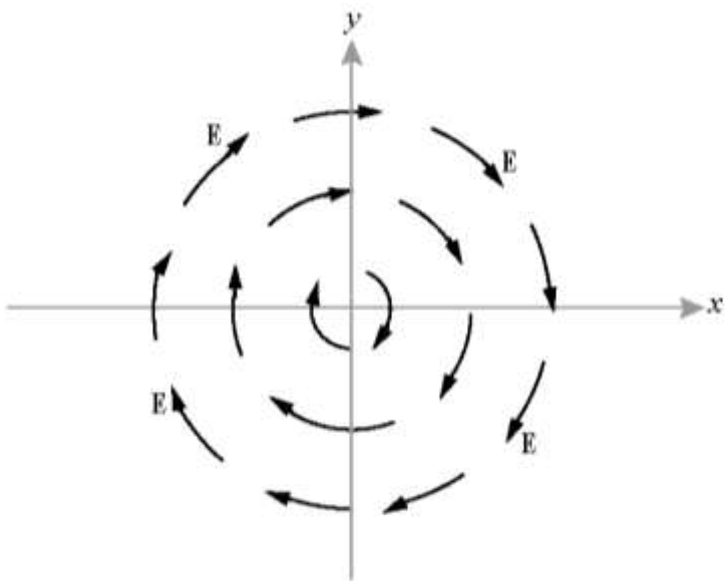
are these in rectangular, cylindrical, or spherical coordinates?



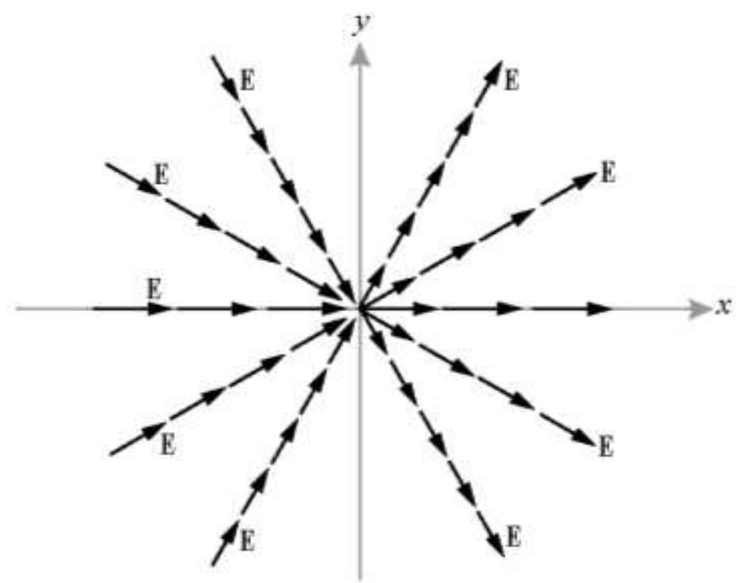
P2.14a: $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$



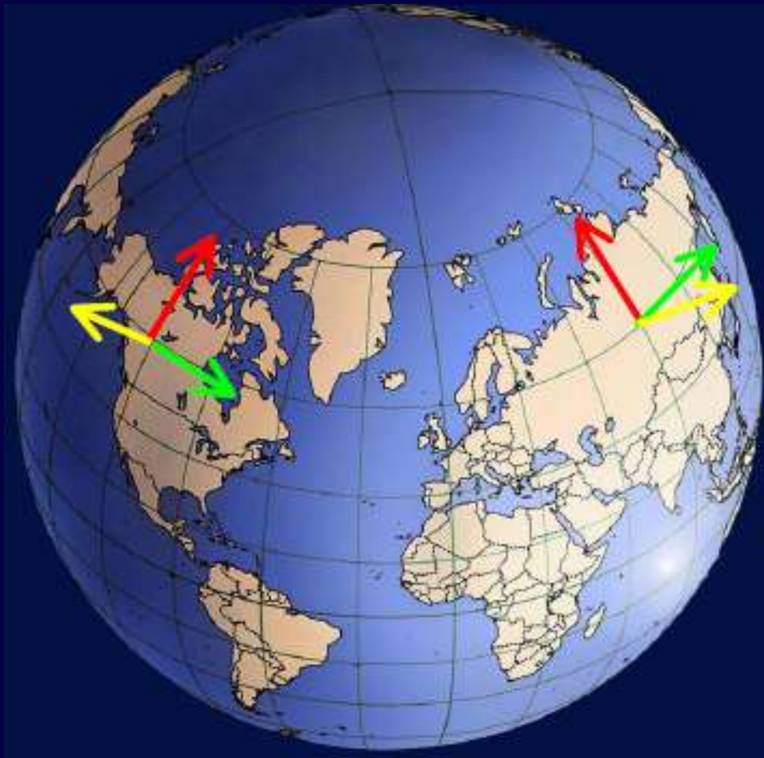
P2.14c: $\mathbf{E}_3 = \hat{\mathbf{y}}(1/x)$



P2.14b: $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$



P2.14d: $\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$



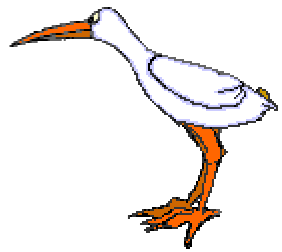
Position coordinates are *different* from basis vectors, and so are their conversion between coordinate systems.

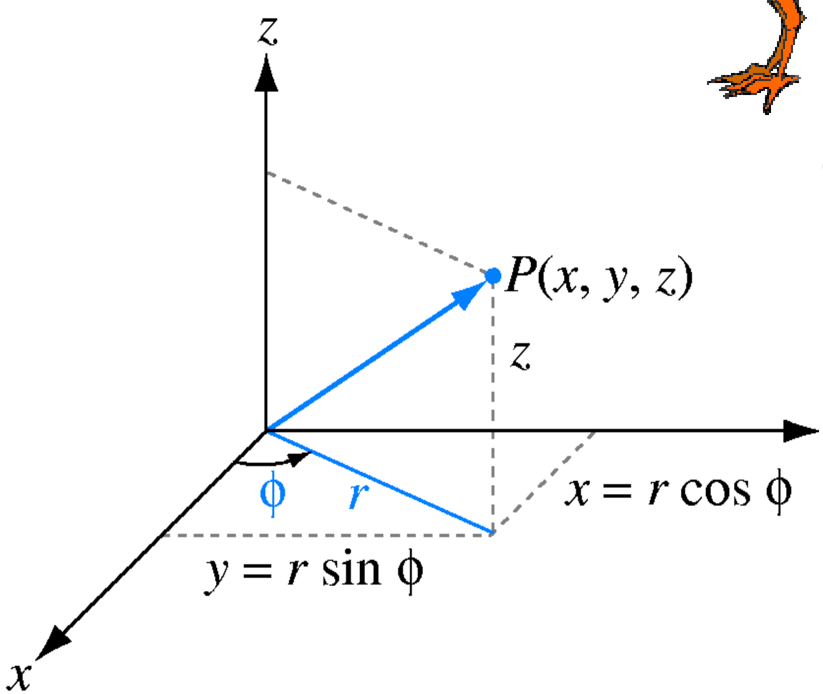
(you likely experienced the *former* in previous courses ☺)

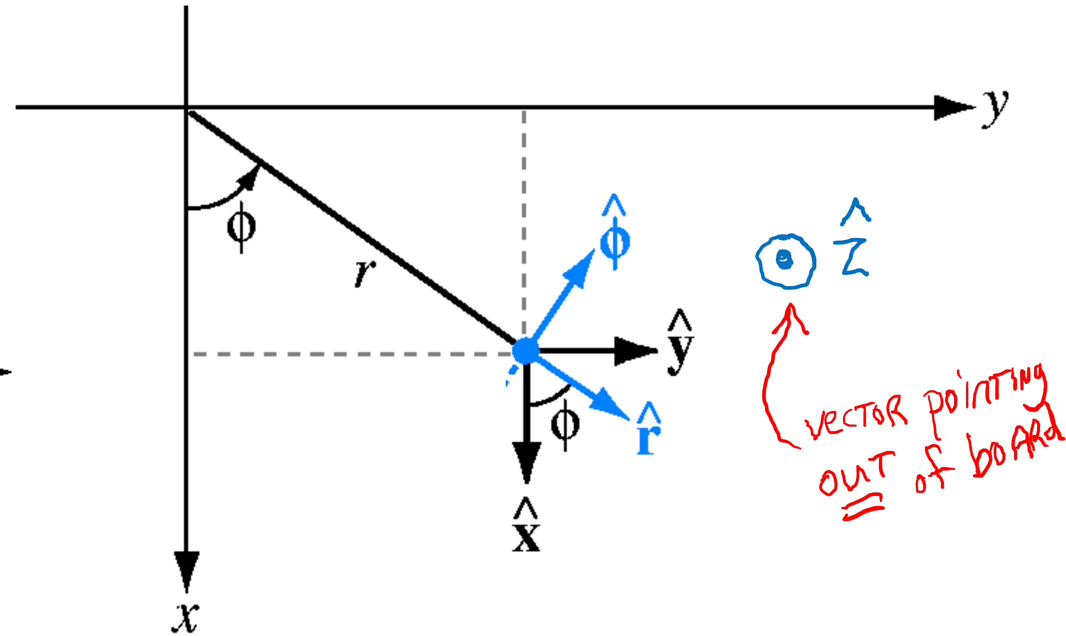
Distance between points is most "easily" calculated in the Cartesian (a.k.a Rectangular) coordinate system.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A \rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}$ =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

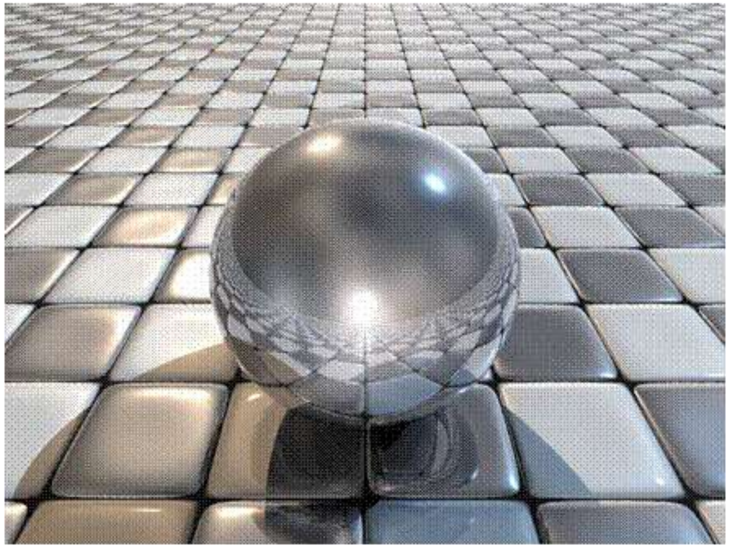
A vector pointing in to the board is represented by ⊗

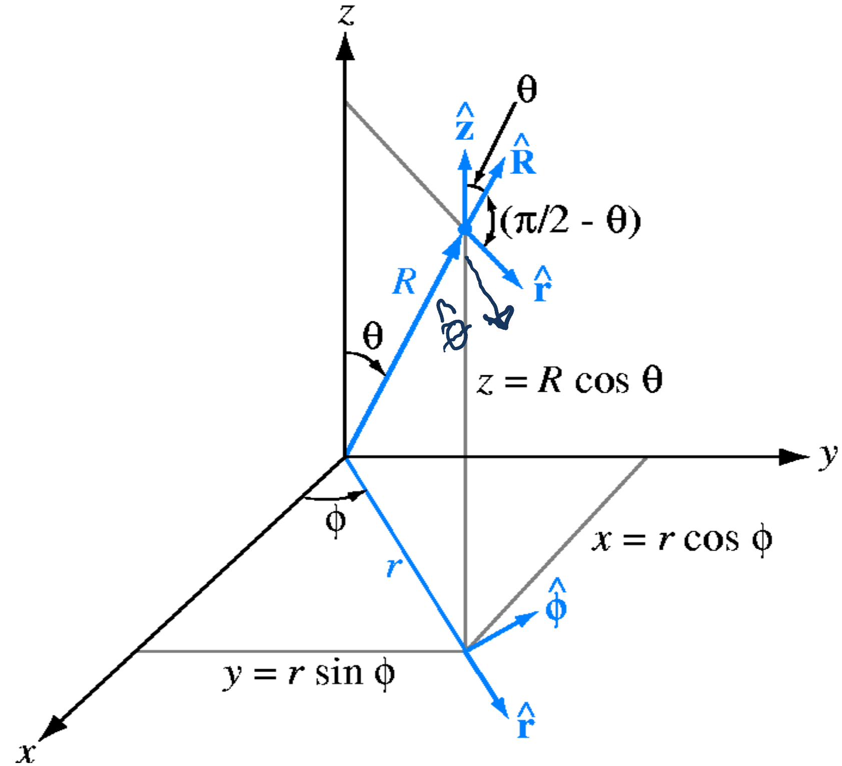






Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$





Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}\left[\sqrt[+]{x^2+y^2}/z\right]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cylindrical to spherical	$R = \sqrt{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Might not need these all too often…

…however, you will learn some additional field operators next:

* **Gradient** (for scalar fields)

* **Divergence** (for vector fields)

* **Curl** (for vector fields)

Problem 3.30 Transform the following vectors into cylindrical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}(x+y)$ at $P_1(1,2,3)$,

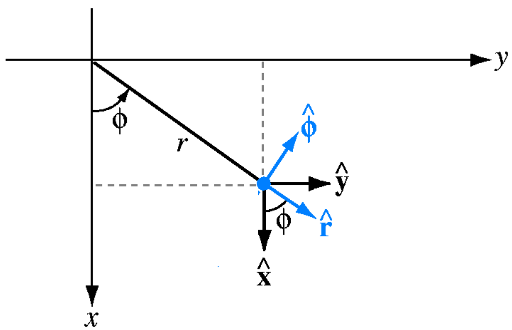
(b) $\mathbf{B} = \hat{\mathbf{x}}(y-x)+\hat{\mathbf{y}}(x-y)$ at $P_2(1,0,2)$,

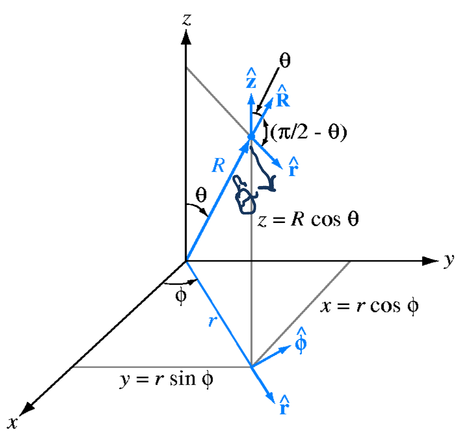
(c) $\mathbf{C} = \hat{\mathbf{x}}y^2/(x^2+y^2)-\hat{\mathbf{y}}x^2/(x^2+y^2)+\hat{\mathbf{z}}4$ at $P_3(1,-1,2)$,

(d) $\mathbf{D} = \hat{\mathbf{R}}\sin\theta+\hat{\boldsymbol{\theta}}\cos\theta+\hat{\boldsymbol{\phi}}\cos^2\phi$ at $P_4(2,\pi/2,\pi/4)$,

(e) $\mathbf{E} = \hat{\mathbf{R}}\cos\phi+\hat{\boldsymbol{\theta}}\sin\phi+\hat{\boldsymbol{\phi}}\sin^2\theta$ at $P_5(3,\pi/2,\pi)$.

what coord. system are we starting in??





Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi+\hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi+A_y\sin\phi$ $A_\phi = -A_x\sin\phi+A_y\cos\phi$ $A_z = A_z$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta+\hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta+A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta-A_\theta\sin\theta$

(a)

$$\mathbf{A} = (\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)(r\cos\phi+r\sin\phi)$$
$$= \hat{\mathbf{r}}r\cos\phi(\cos\phi+\sin\phi)-\hat{\boldsymbol{\phi}}r\sin\phi(\cos\phi+\sin\phi),$$
$$P_1 = (\sqrt{1^2+2^2},\tan^{-1}(2/1),3) = (\sqrt{5},63.4^\circ,3),$$
$$\mathbf{A}(P_1) = (\hat{\mathbf{r}}0.447-\hat{\boldsymbol{\phi}}0.894)\sqrt{5}(.447+.894) = \hat{\mathbf{r}}1.34-\hat{\boldsymbol{\phi}}2.68.$$

(b)

$$\mathbf{B} = (\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)(r\sin\phi-r\cos\phi)+(\hat{\boldsymbol{\phi}}\cos\phi+\hat{\mathbf{r}}\sin\phi)(r\cos\phi-r\sin\phi)$$
$$= \hat{\mathbf{r}}r(2\sin\phi\cos\phi-1)+\hat{\boldsymbol{\phi}}r(\cos^2\phi-\sin^2\phi) = \hat{\mathbf{r}}r(\sin 2\phi-1)+\hat{\boldsymbol{\phi}}r\cos 2\phi,$$
$$P_2 = (\sqrt{1^2+0^2},\tan^{-1}(0/1),2) = (1,0^\circ,2),$$
$$\mathbf{B}(P_2) = -\hat{\mathbf{r}}+\hat{\boldsymbol{\phi}}.$$

(c)

$$\mathbf{C} = (\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)\frac{r^2\sin^2\phi}{r^2}-(\hat{\boldsymbol{\phi}}\cos\phi+\hat{\mathbf{r}}\sin\phi)\frac{r^2\cos^2\phi}{r^2}+\hat{\mathbf{z}}4$$
$$= \hat{\mathbf{r}}\sin\phi\cos\phi(\sin\phi-\cos\phi)-\hat{\boldsymbol{\phi}}(\sin^3\phi+\cos^3\phi)+\hat{\mathbf{z}}4,$$
$$P_3 = (\sqrt{1^2+(-1)^2},\tan^{-1}(-1/1),2) = (\sqrt{2},-45^\circ,2),$$
$$\mathbf{C}(P_3) = \hat{\mathbf{r}}0.707+\hat{\mathbf{z}}4.$$

(d)

$$\mathbf{D} = (\hat{\mathbf{r}}\sin\theta+\hat{\mathbf{z}}\cos\theta)\sin\theta+(\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}\sin\theta)\cos\theta+\hat{\boldsymbol{\phi}}\cos^2\phi = \hat{\mathbf{r}}+\hat{\boldsymbol{\phi}}\cos^2\phi,$$
$$P_4 = (2\sin(\pi/2),\pi/4,2\cos(\pi/2)) = (2,45^\circ,0),$$
$$\mathbf{D}(P_4) = \hat{\mathbf{r}}+\hat{\boldsymbol{\phi}}\tfrac{1}{2}.$$

(e)

$$\mathbf{E} = (\hat{\mathbf{r}}\sin\theta+\hat{\mathbf{z}}\cos\theta)\cos\phi+(\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}\sin\theta)\sin\phi+\hat{\boldsymbol{\phi}}\sin^2\theta,$$
$$P_5 = \left(3,\frac{\pi}{2},\pi\right),$$
$$\mathbf{E}(P_5) = \left(\hat{\mathbf{r}}\sin\frac{\pi}{2}+\hat{\mathbf{z}}\cos\frac{\pi}{2}\right)\cos\pi+\left(\hat{\mathbf{r}}\cos\frac{\pi}{2}-\hat{\mathbf{z}}\sin\frac{\pi}{2}\right)\sin\pi+\hat{\boldsymbol{\phi}}\sin^2\frac{\pi}{2} = -\hat{\mathbf{r}}+\hat{\boldsymbol{\phi}}.$$

T-Line Example…

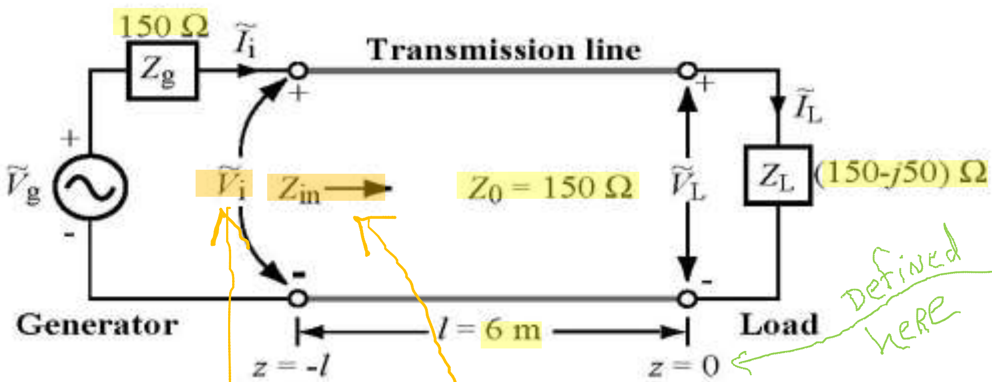
Problem 2.22 A 6-m section of 150-Ω lossless line is driven by a source with

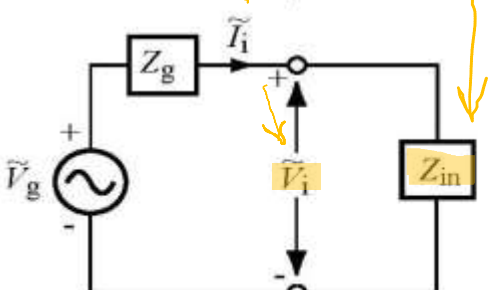
$$v_g(t) = 5\cos(8\pi \times 10^7 t - 30^\circ) \quad \text{(V)}$$

$\Rightarrow 2\pi f \Rightarrow f = 40\text{ MHz}$

and $Z_g = 150\ \Omega$. If the line, which has a relative permittivity $\varepsilon_r = 2.25$, is terminated in a load $Z_L = (150 - j50)\ \Omega$, find

(a) λ on the line,
(b) the reflection coefficient at the load,
(c) the input impedance,
(d) the input voltage $\widetilde{V}_i$,
(e) the time-domain input voltage $v_i(t)$.





Solution:

$$v_g(t) = 5\cos(8\pi \times 10^7 t - 30^\circ)\ \text{V},$$
$$\widetilde{V}_g = 5e^{-j30^\circ}\ \text{V}.$$

(a)

$$u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8 \quad \text{(m/s)},$$

more on ε_R later ☺

$$\lambda = \frac{u_p}{f} = \frac{2\pi u_p}{\omega} = \frac{2\pi \times 2 \times 10^8}{8\pi \times 10^7} = 5\ \text{m},$$

$$\beta = \frac{\omega}{u_p} = \frac{8\pi \times 10^7}{2 \times 10^8} = 0.4\pi \quad \text{(rad/m)},$$

$$\beta l = 0.4\pi \times 6 = 2.4\pi \quad \text{(rad)}.$$

$\Rightarrow l = 1.2\lambda$

can only get away with this when it's lossless

Since this exceeds 2π (rad), we can subtract 2π, which leaves a remainder $\beta l = 0.4\pi$ (rad).

$\rightarrow l_{eff} = 0.2\lambda$

(b) $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{150 - j50 - 150}{150 - j50 + 150} = \dfrac{-j50}{300 - j50} = 0.16e^{-j80.54^\circ}$.

(c)

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] \qquad Z_0\left[\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right]$$

$$= 150\left[\frac{(150 - j50) + j150\tan(0.4\pi)}{150 + j(150 - j50)\tan(0.4\pi)}\right] = (115.70 + j27.42)\ \Omega.$$

(d)

Phasors and Voltage Dividers

$$\widetilde{V}_i = \frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}} = \frac{5e^{-j30^\circ}(115.7 + j27.42)}{150 + 115.7 + j27.42}$$

$$= 5e^{-j30^\circ}\left(\frac{115.7 + j27.42}{265.7 + j27.42}\right)$$

$$= 5e^{-j30^\circ} \times 0.44e^{j7.44^\circ} = 2.2e^{-j22.56^\circ} \quad \text{(V)}.$$

(e)

$$v_i(t) = \Re\mathfrak{e}[\widetilde{V}_i e^{j\omega t}] = \Re\mathfrak{e}[2.2e^{-j22.56^\circ} e^{j\omega t}] = 2.2\cos(8\pi \times 10^7 t - 22.56^\circ)\ \text{V}.$$

Phasors back to time domain

“Mathematics is the art of giving the same name to different things.”

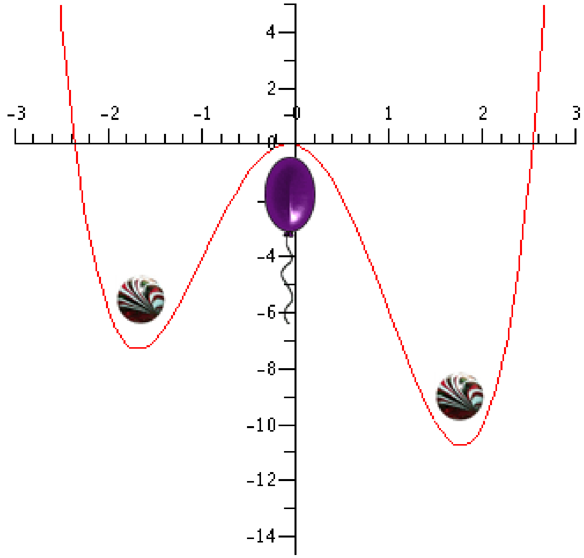
-J. H. Poincare

http://en.wikipedia.org/wiki/Henri_Poincar%C3%A9

"Mathematics is a game played according to certain simple rules with meaningless marks on paper.“

- David Hilbert

http://en.wikipedia.org/wiki/David_Hilbert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-2
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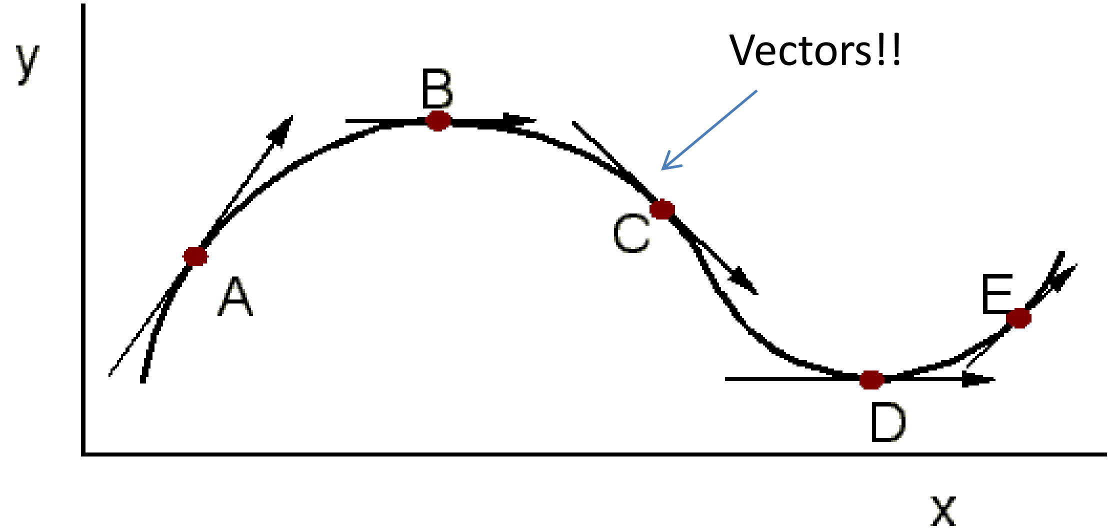
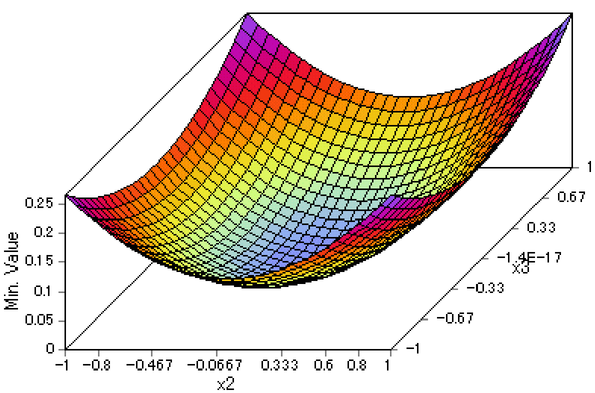
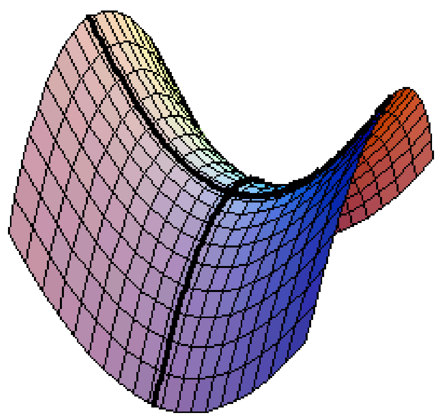

y
Vectors!!
B
A
C
E
D
x

2-D


Min. Value
0.25
0.2
0.15
0.1
0.05
0
-1 -0.8 -0.467 -0.0667 0.333 0.6 0.8 1
x2
x3
1
0.67
0.33
-1.4E-17
-0.33
-0.67
-1



3-D


FROM THE DIRECTOR OF "TITANIC"
AVATAR

Gradient

- Max/Min
- Optimization
- Physical “flows”

del operator

Cartesian

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

other...

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

Gradient

scalar field $V(x,y,z)$ $\Rightarrow$

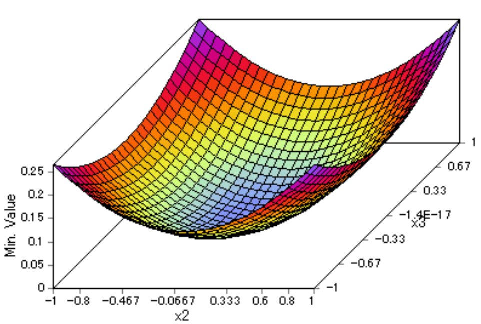
$$\text{grad } V \equiv \nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$

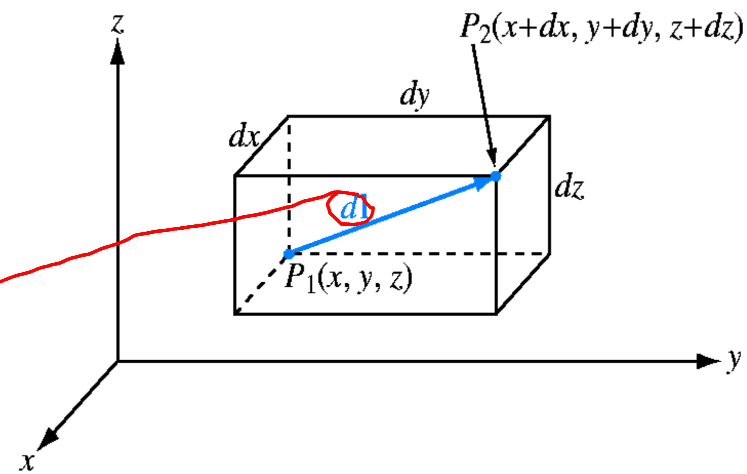
vector field

Directional Derivative

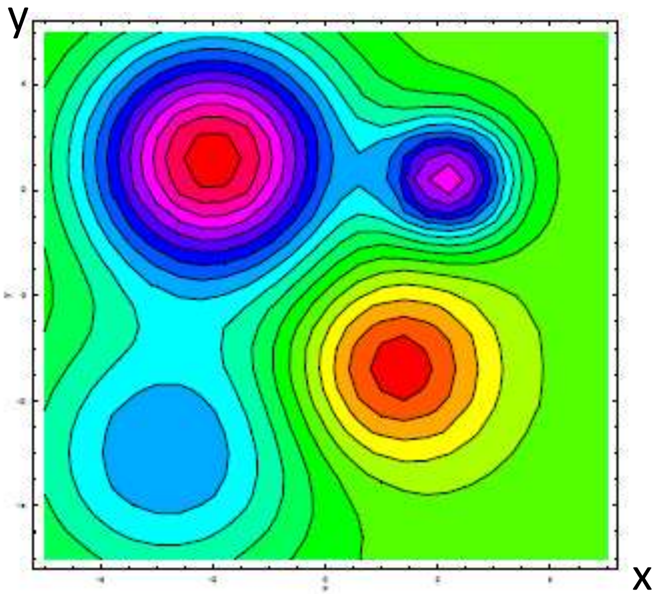
$$\frac{\partial V}{\partial l} = \nabla V \cdot \underbrace{\hat{a}_l}_{\text{unit vector of } d\vec{l}}$$

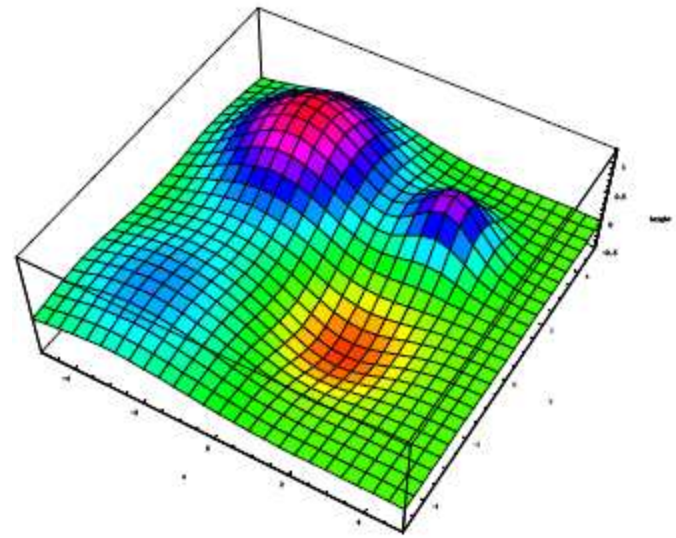
unit vector of $d\vec{l}$



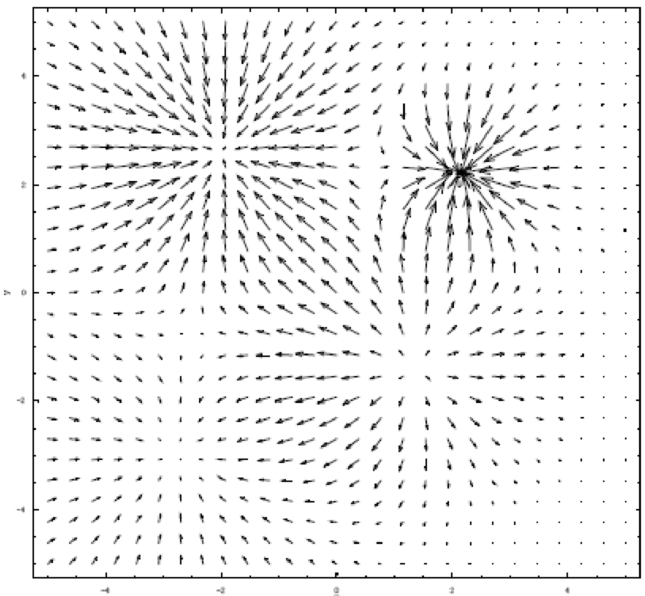


Height, h(x,y) … a *scalar* field

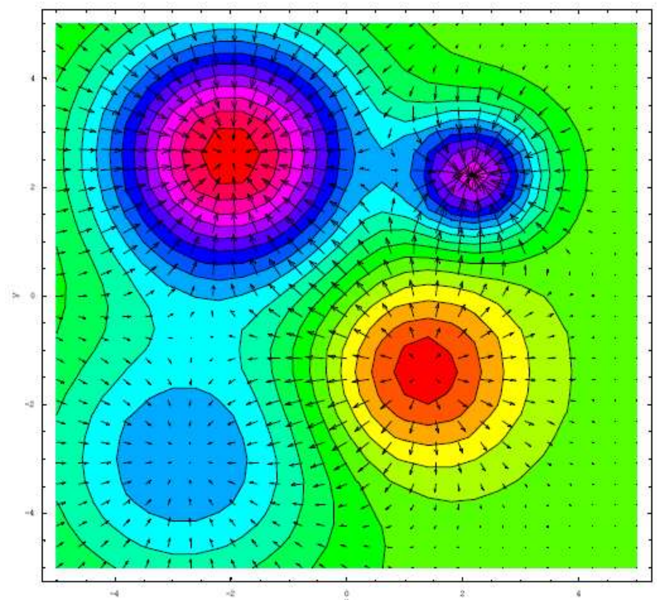




Gradient of h(x,y), $\nabla h(x,y)$ … a *vector* field



Overlay



Problem 3.32 Find the gradient of the following scalar functions:

(a) $T = 3/(x^2 + z^2)$,
(b) $V = xy^2z^4$,
(c) $U = z\cos\phi/(1 + r^2)$,
(d) $W = e^{-R}\sin\theta$,
(e) $S = 4x^2e^{-z} + y^3$,
(f) $N = r^2\cos^2\phi$,
(g) $M = R\cos\theta\sin\phi$.

<u>Figure out your coordinate system!!</u>

$$\boxed{\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}}$$

$$\boxed{\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}}$$

$$\boxed{\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}}$$

(a) From Eq. (3.72),

$$\nabla T = -\hat{\mathbf{x}}\frac{6x}{(x^2+z^2)^2} - \hat{\mathbf{z}}\frac{6z}{(x^2+z^2)^2}.$$

(b) From Eq. (3.72),

$$\nabla V = \hat{\mathbf{x}}y^2z^4 + \hat{\mathbf{y}}2xyz^4 + \hat{\mathbf{z}}4xy^2z^3.$$

(c) From Eq. (3.82),

$$\nabla U = -\hat{\mathbf{r}}\frac{2rz\cos\phi}{(1+r^2)^2} - \hat{\boldsymbol{\phi}}\frac{z\sin\phi}{r(1+r^2)} + \hat{\mathbf{z}}\frac{\cos\phi}{1+r^2}.$$

(d) From Eq. (3.83),

$$\nabla W = -\hat{\mathbf{R}}e^{-R}\sin\theta + \hat{\boldsymbol{\theta}}(e^{-R}/R)\cos\theta.$$

(e) From Eq. (3.72),

$$S = 4x^2e^{-z} + y^3,$$

$$\nabla S = \hat{\mathbf{x}}\frac{\partial S}{\partial x} + \hat{\mathbf{y}}\frac{\partial S}{\partial y} + \hat{\mathbf{z}}\frac{\partial S}{\partial z} = \hat{\mathbf{x}}8xe^{-z} + \hat{\mathbf{y}}3y^2 - \hat{\mathbf{z}}4x^2e^{-z}.$$

(f) From Eq. (3.82),

$$N = r^2\cos^2\phi,$$

$$\nabla N = \hat{\mathbf{r}}\frac{\partial N}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial N}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial N}{\partial z} = \hat{\mathbf{r}}2r\cos^2\phi - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi.$$

(g) From Eq. (3.83),

$$M = R\cos\theta\sin\phi,$$

$$\nabla M = \hat{\mathbf{R}}\frac{\partial M}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial M}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial M}{\partial \phi} = \hat{\mathbf{R}}\cos\theta\sin\phi - \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{\tan\theta}.$$

Problem 3.36 For the scalar function $T = \frac{1}{2}e^{-r/5}\cos\phi$, determine its directional derivative along the radial direction $\hat{\mathbf{r}}$ and then evaluate it at $P(2, \pi/4, 3)$.

Cylindrical

$$T = \frac{1}{2}e^{-r/5}\cos\phi,$$

$$\nabla T = \hat{\mathbf{r}}\frac{\partial T}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} = -\hat{\mathbf{r}}\frac{e^{-r/5}\cos\phi}{10} - \hat{\boldsymbol{\phi}}\frac{e^{-r/5}\sin\phi}{2r},$$

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{r}} = -\frac{e^{-r/5}\cos\phi}{10},$$

$$\left.\frac{dT}{dl}\right|_{(2,\pi/4,3)} = -\frac{e^{-2/5}\cos\frac{\pi}{4}}{10} = -4.74 \times 10^{-2}.$$

Directional Derivative in $\hat{r}$ direction.

Example:

Find the gradient of the scalar field $a = a_0 e^{-x}\sin(\pi y)$ and determine its value at (1,2,0).

$$\nabla a = \hat{x}\frac{\partial a}{\partial x} + \hat{y}\frac{\partial a}{\partial y} + \hat{z}\frac{\partial a}{\partial z} =$$
$$= -\hat{x}a_0 e^{-x}\sin(\pi y) + \hat{y}\pi a_0 e^{-x}\cos(\pi y) =$$
$$= [-\hat{x}\sin(\pi y) + \hat{y}\pi\cos(\pi y)]a_0 e^{-x}$$

$$\nabla a(1,2,0) = [-\hat{x}\sin(2\pi) + \hat{y}\pi\cos(2\pi)]a_0 e^{-1} = \hat{y}\frac{a_0\pi}{e}$$

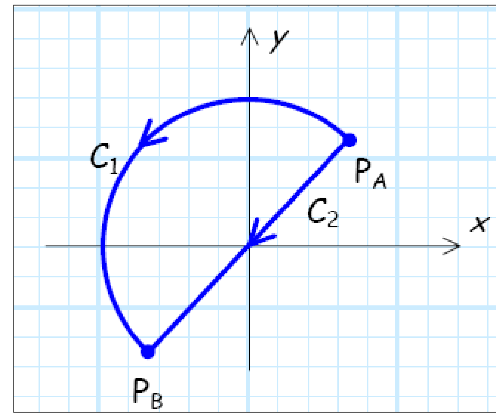
Conservative Field

1. A conservative vector field can always be expressed as the **gradient** of a **scalar** field.

$$\mathbf{C}(\bar{\mathbf{r}}) = \nabla g(\bar{\mathbf{r}})$$

2. The gradient of **any** scalar field is therefore a conservative vector field.

2.5 Not all vector fields are conservative!!

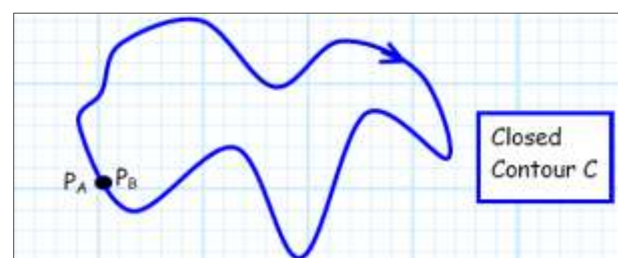


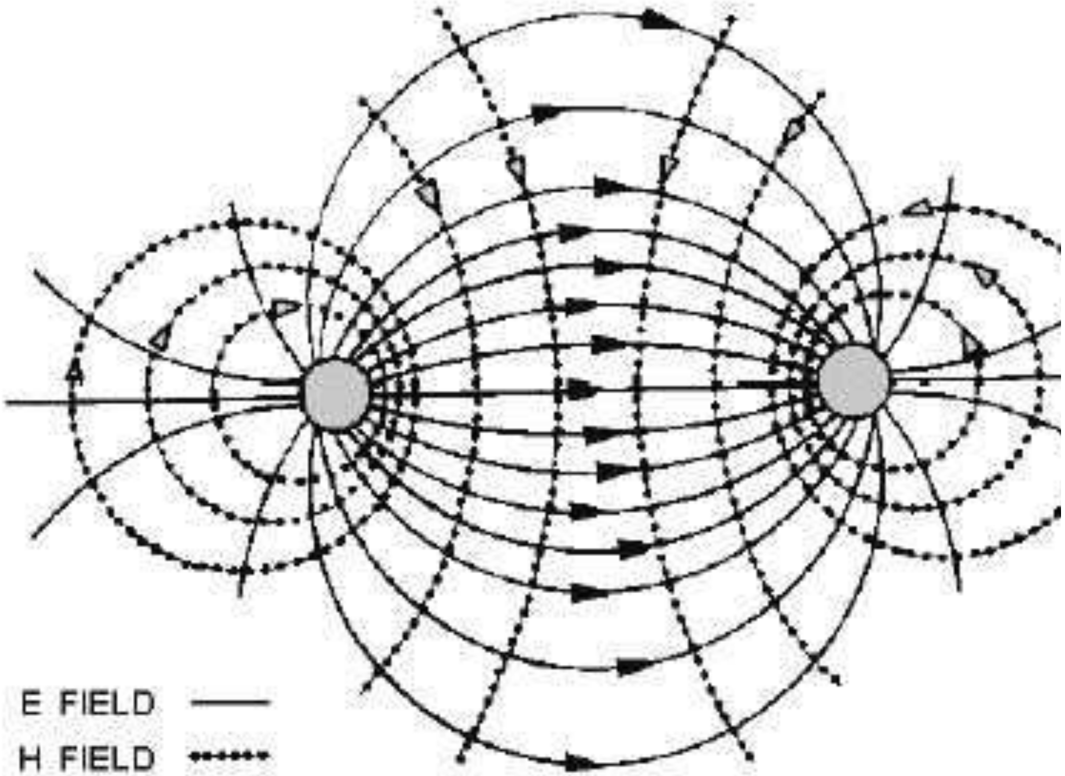
3. Integration over an **open** contour is dependent **only** on the value of scalar field $g(\bar{r})$ at the beginning and ending points of the contour (i.e., integration is **path independent**).

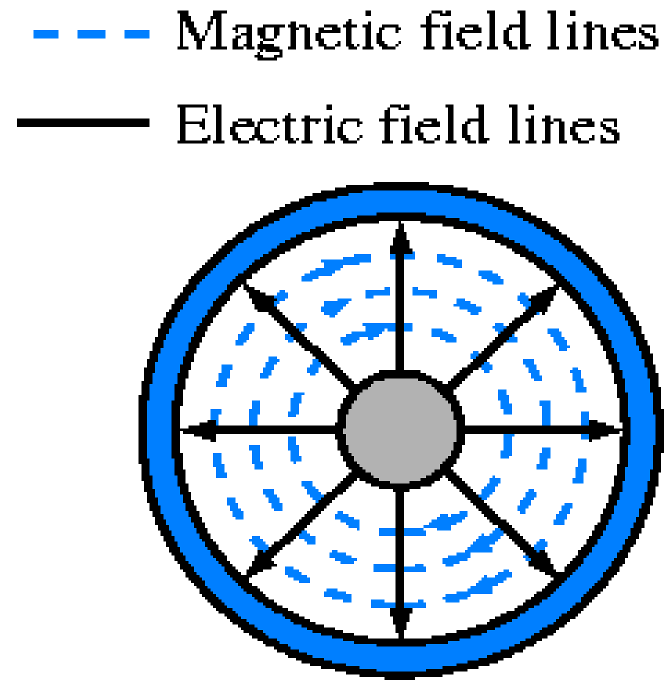
$$\int_C C(\bar{r})\cdot\overline{d\ell} = \int_C \nabla g(\bar{r})\cdot\overline{d\ell}$$
$$= g(\bar{r}=\bar{r}_B) - g(\bar{r}=\bar{r}_A)$$

3.5 Path integration is fun ??!!

4. Integration of a conservative vector field over any **closed** contour is always equal to **zero**.





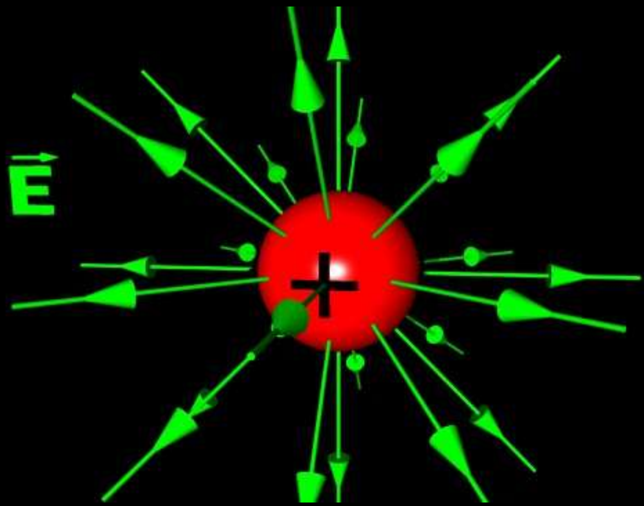


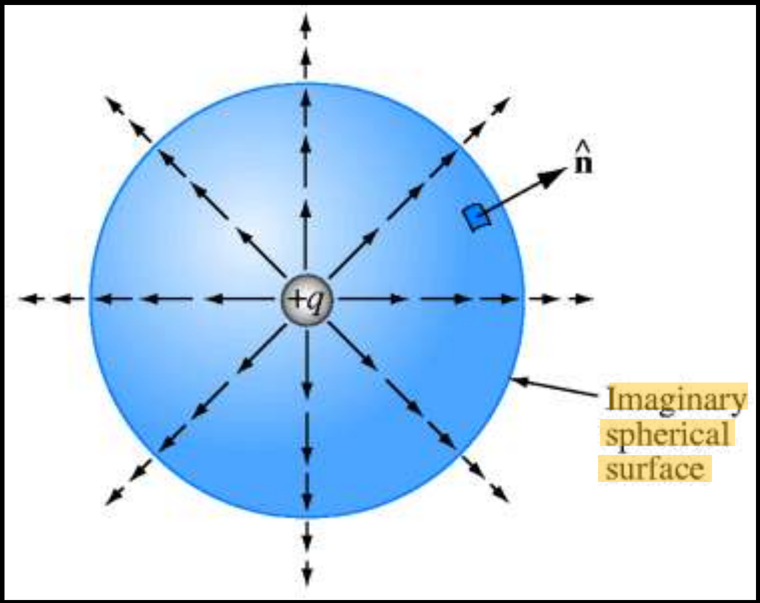
Cross section

$$\vec{E} = -\nabla V \qquad \Rightarrow \qquad \nabla \times \vec{E} = \nabla \times \left(-\nabla V \right) = 0$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

Electric Field



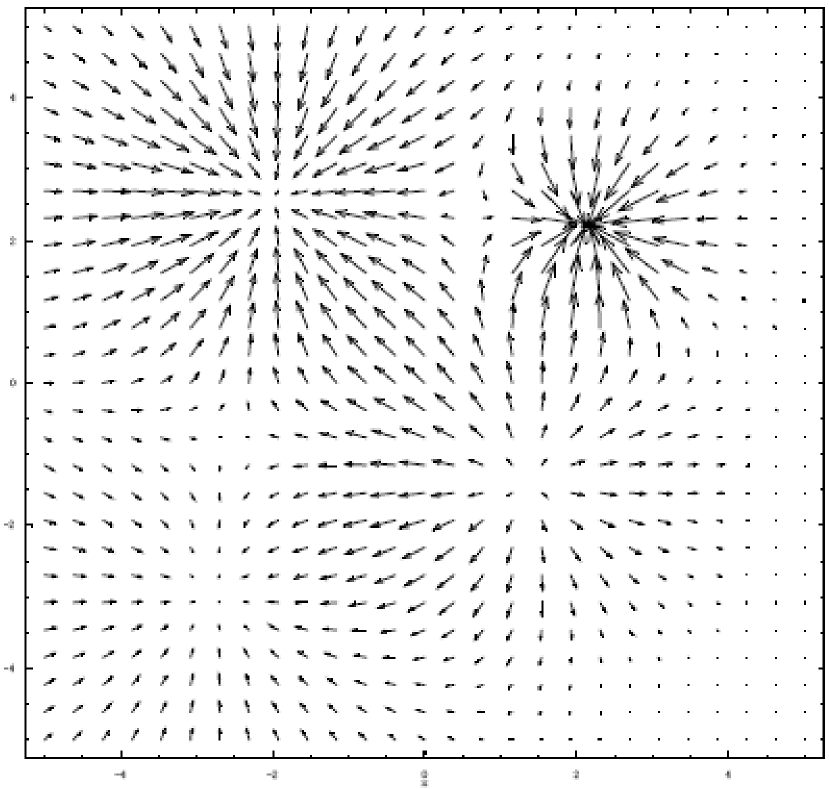


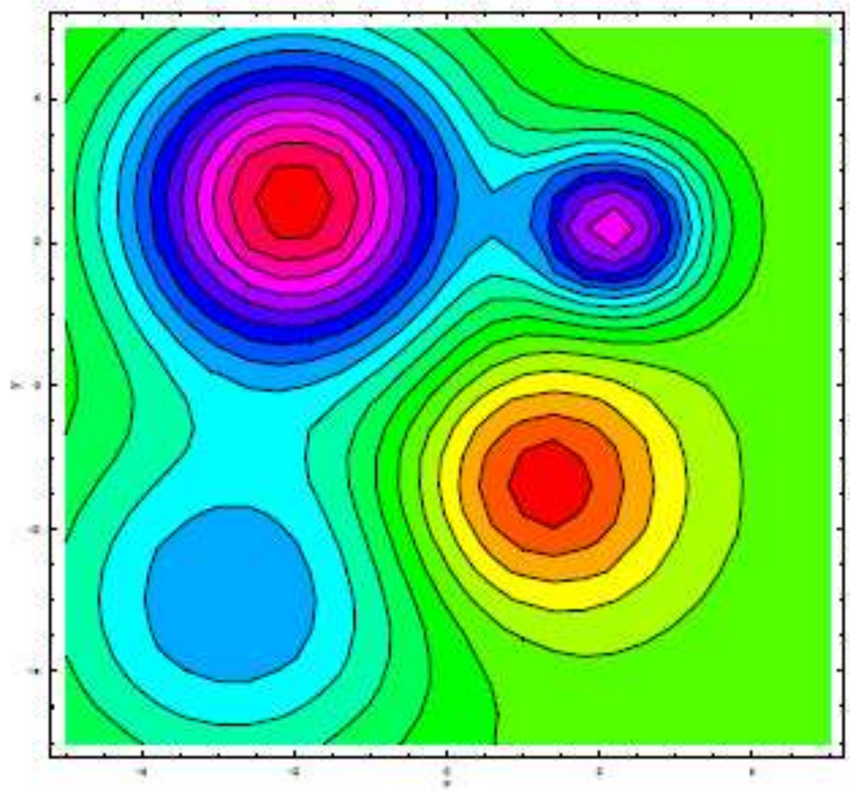
The **mathematical** definition of divergence is:

$$\nabla \cdot \mathbf{A}(\bar{r}) = \lim_{\Delta v \to 0} \frac{\oint\!\!\oint_S \mathbf{A}(\bar{r}) \cdot \overline{ds}}{\Delta v}$$

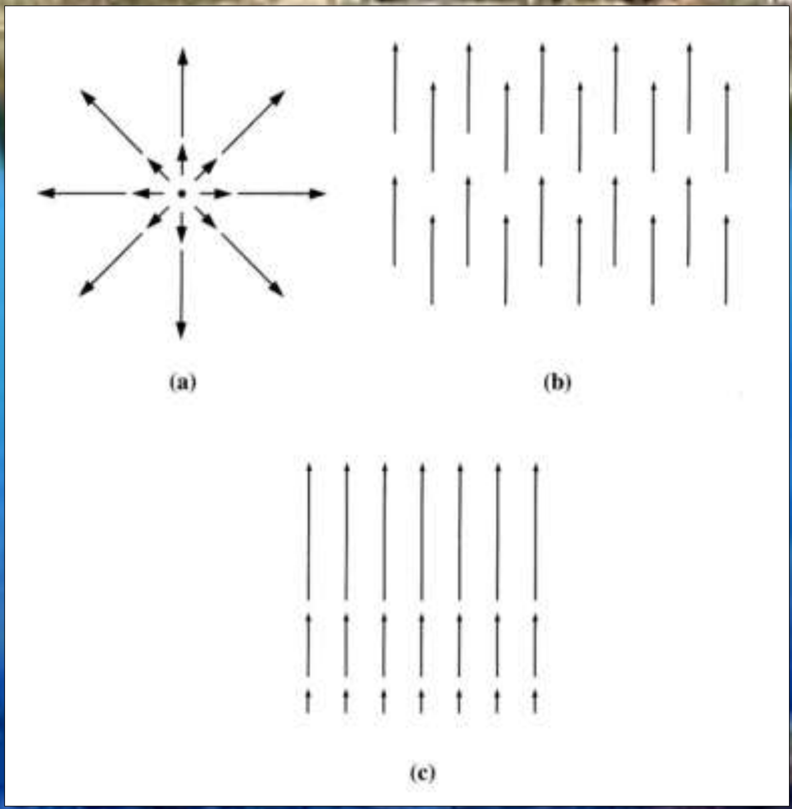
where the surface S is a **closed** surface that **completely** surrounds a **very small** volume Δv at point $\bar{r}$, and where $\overline{ds}$ points **outward** from the closed surface.

If you know gradient everywhere, can you tell me what the scalar field is???





Divergence



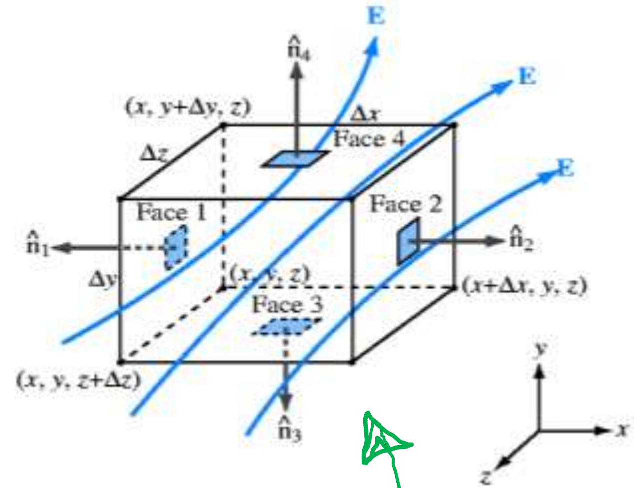
Pine needles in a pond…

Vector fields have spatial variability

The **mathematical** definition of divergence is:

$$\nabla \cdot \mathbf{A}(\bar{r}) = \lim_{\Delta v \to 0} \frac{\oint\!\!\oint_S \mathbf{A}(\bar{r}) \cdot \overline{ds}}{\Delta v}$$

where the surface S is a **closed** surface that **completely** surrounds a **very small** volume Δv at point $\bar{r}$, and where $\overline{ds}$ points **outward** from the closed surface.



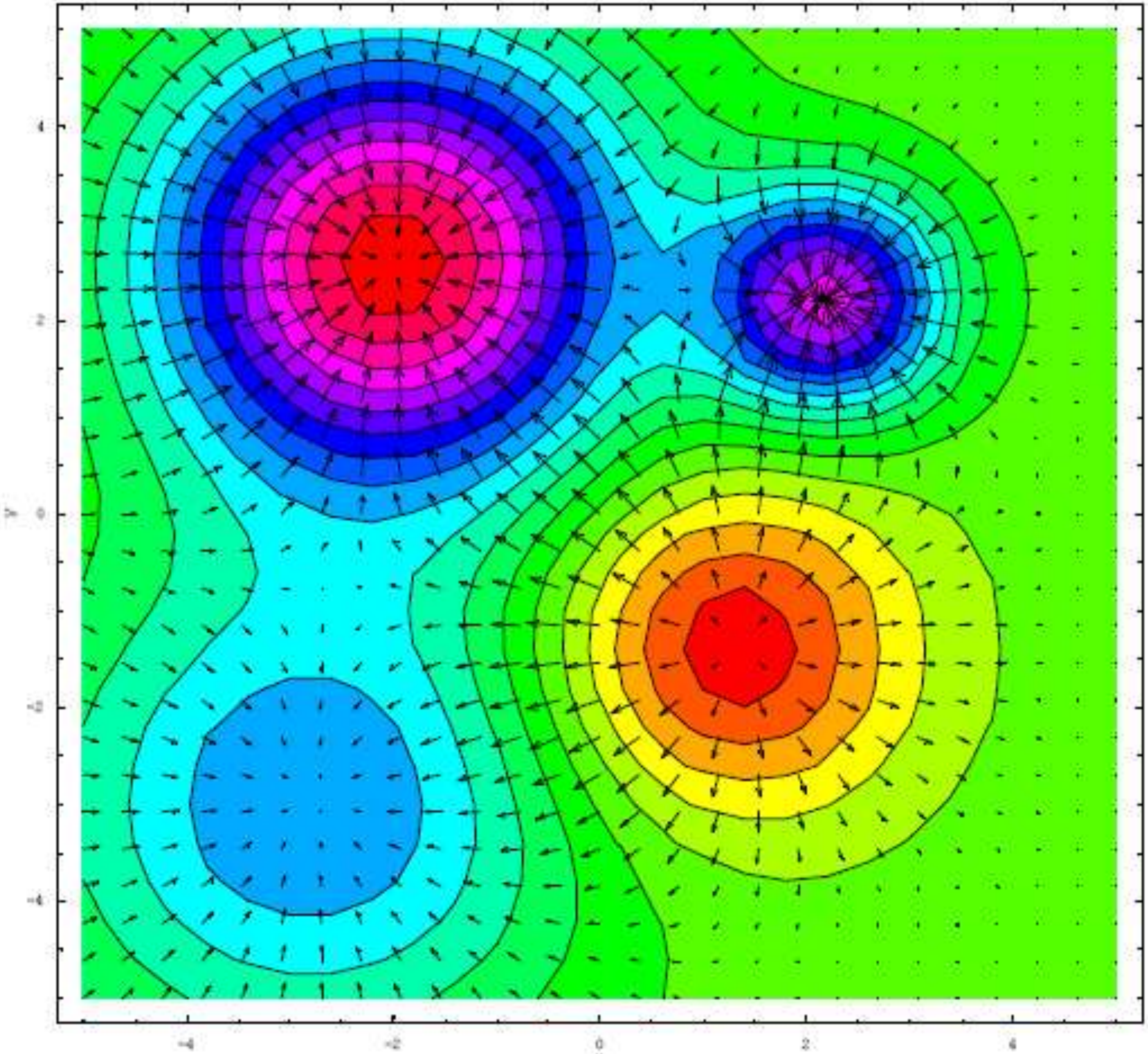
$\vec{A}$ is a vector field

div $\vec{A}$ = $\nabla \cdot \vec{A}$ is a scalar field !

$$\boxed{\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

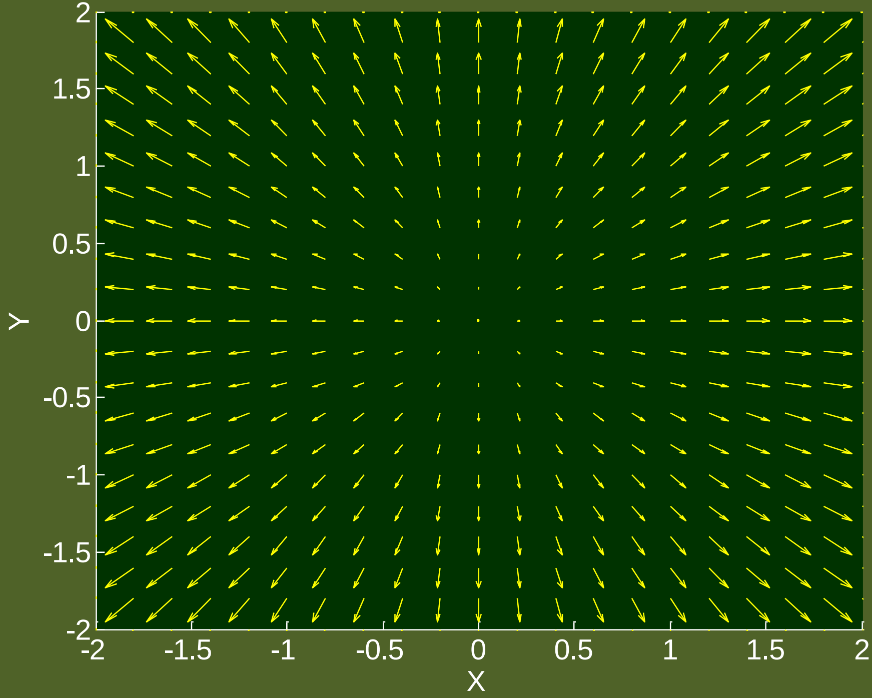
$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}}$$

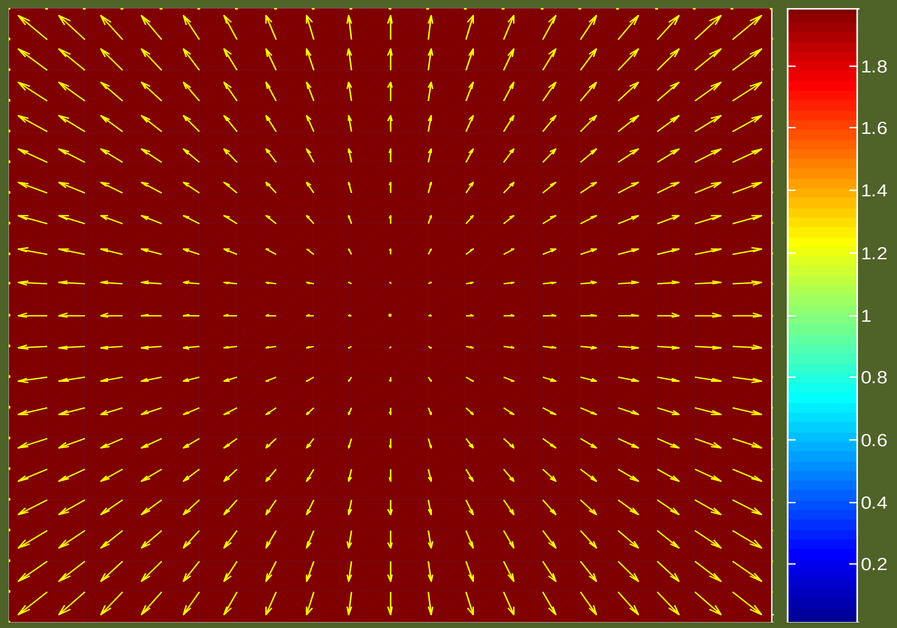


-4
-2
0
2
4
x
y

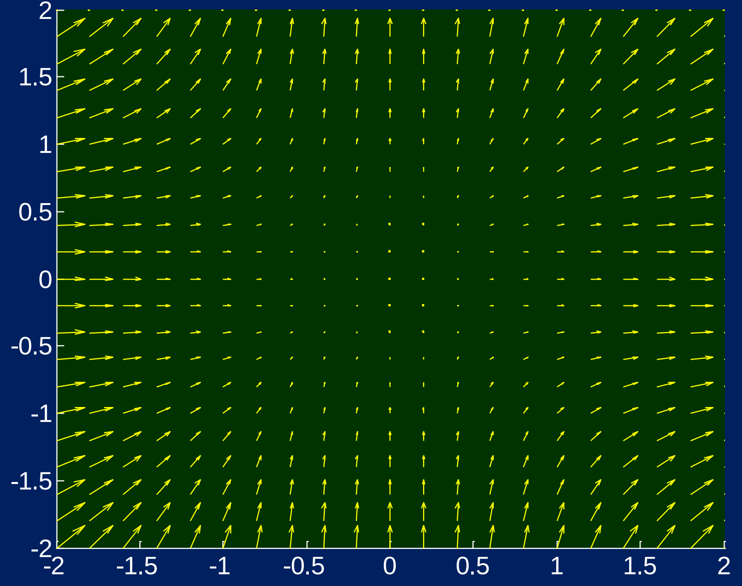
$$\vec{\mathbf{V}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

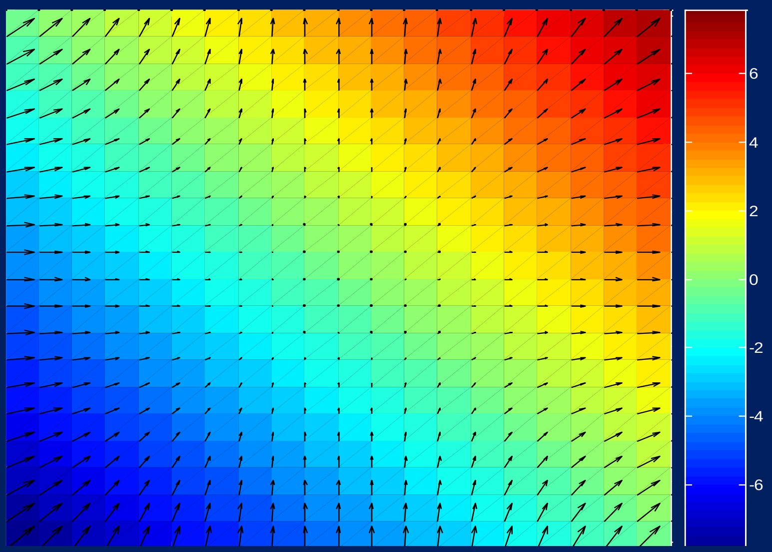
$$\nabla \cdot \vec{\mathbf{V}} = 2$$





$$\vec{\mathbf{V}} = x^2\hat{\mathbf{x}} + y^2\hat{\mathbf{y}} \qquad\qquad \nabla \cdot \vec{\mathbf{V}} = 2x + 2y$$





Divergence Theorem

(or Gauss', or Green's, etc.)

or dV

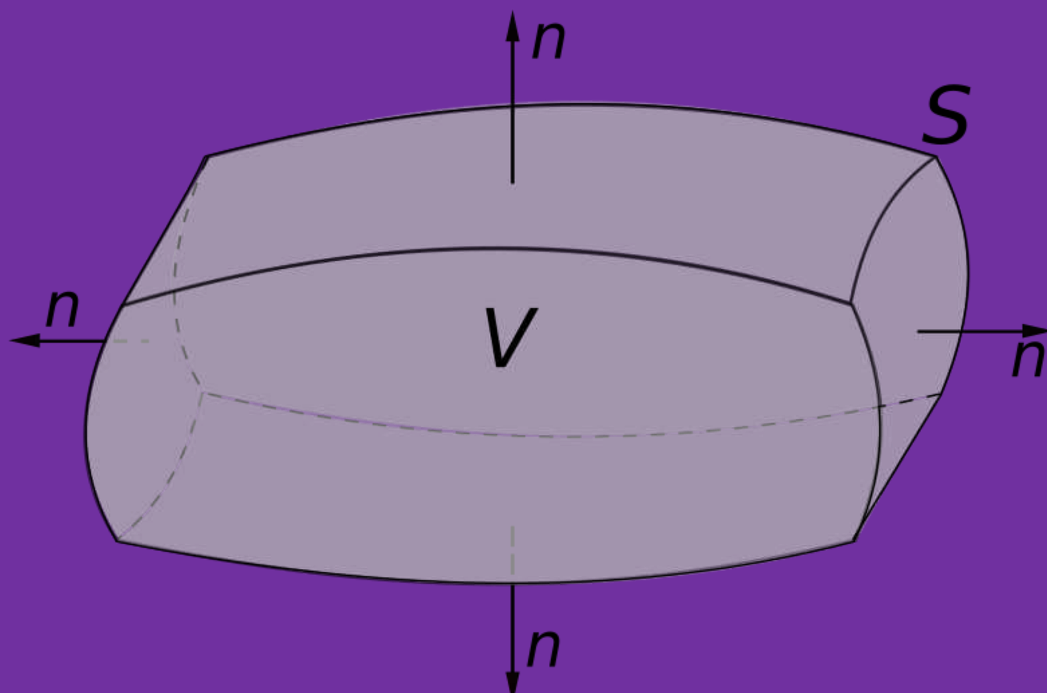
or $d\vec{s}$

$$\int_V \left(\nabla \cdot \vec{\mathbf{V}}\right) d\tau = \oint_S \vec{\mathbf{V}} \cdot d\vec{\mathbf{a}}$$

Volume

Closed surface containing volume V

normal to surface

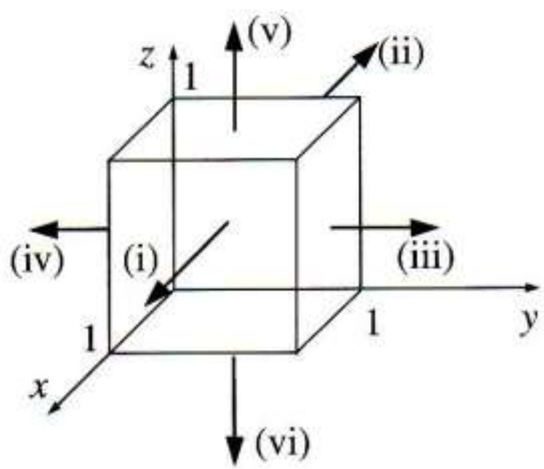


BTW… variables are, well, variable !! ☺

Check the divergence theorem using the function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

and the unit cube situated at the origin



$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Solution: In this case

$$\nabla \cdot \mathbf{v} = 2(x + y),$$

and

$$\int_{\mathcal{V}} 2(x + y)\,d\tau = 2\int_0^1\int_0^1\int_0^1 (x + y)\,dx\,dy\,dz,$$

$$\int_0^1 (x + y)\,dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2} + y)\,dy = 1, \quad \int_0^1 1\,dz = 1.$$

Evidently,

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v}\,d\tau = 2.$$

So much for the left side of the divergence theorem. To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1\int_0^1 y^2\,dy\,dz = \tfrac{1}{3}.$$

(ii) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1\int_0^1 y^2\,dy\,dz = -\tfrac{1}{3}.$$

(iii) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1\int_0^1 (2x + z^2)\,dx\,dz = \tfrac{4}{3}.$$

(iv) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\tfrac{1}{3}.$$

(v) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1\int_0^1 2y\,dx\,dy = 1.$$

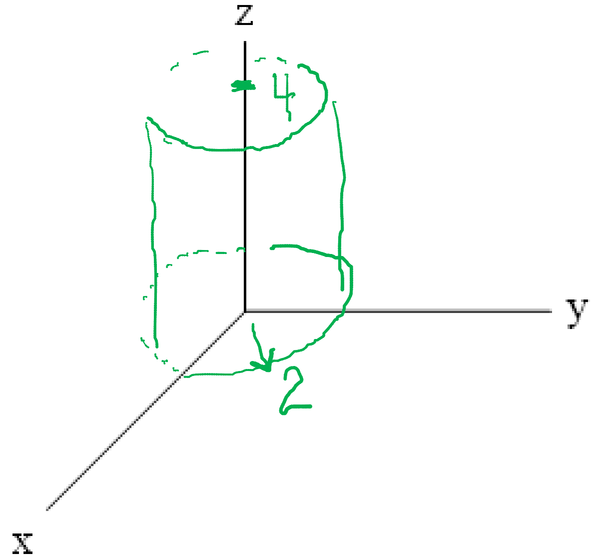
(vi) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1\int_0^1 0\,dx\,dy = 0.$$

So the total flux is:

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{1}{3} + 1 + 0 = 2,$$

as expected.

Problem 3.40 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.



Solution:

$$
\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{s} &= \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (-\hat{\mathbf{z}}r\,dr\,d\phi) \right) \Big|_{z=0} \\
&\quad + \int_{\phi=0}^{2\pi} \int_{z=0}^{4} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{r}}r\,d\phi\,dz) \right) \Big|_{r=2} \\
&\quad + \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{z}}r\,dr\,d\phi) \right) \Big|_{z=4} \\
&= 0 + \int_{\phi=0}^{2\pi} \int_{z=0}^{4} 10e^{-2}2\,d\phi\,dz + \int_{r=0}^{2} \int_{\phi=0}^{2\pi} -12r\,dr\,d\phi \\
&= 160\pi e^{-2} - 48\pi \approx -82.77,
\end{aligned}
$$

$$
\begin{aligned}
\iiint \nabla \cdot \mathbf{E}\,dv &= \int_{z=0}^{4} \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left(\frac{10e^{-r}(1-r)}{r} - 3 \right) r\,d\phi\,dr\,dz \\
&= 8\pi \int_{r=0}^{2} \left(10e^{-r}(1-r) - 3r \right) dr \\
&= 8\pi \left(-10e^{-r} + 10e^{-r}(1+r) - \frac{3r^2}{2} \right) \Bigg|_{r=0}^{2} \\
&= 160\pi e^{-2} - 48\pi \approx -82.77.
\end{aligned}
$$

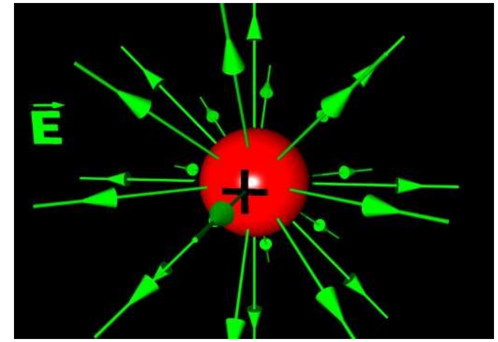
$$
\begin{aligned}
d\mathbf{s}_r &= \hat{\mathbf{r}}\,r\,d\phi\,dz \\
d\mathbf{s}_\phi &= \hat{\boldsymbol{\phi}}\,dr\,dz \\
d\mathbf{s}_z &= \hat{\mathbf{z}}\,r\,dr\,d\phi
\end{aligned}
$$

$$r\,dr\,d\phi\,dz$$

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

Vector fields have spatial variability
(EXAMPLE)

$$\vec{E} = \frac{1}{R^2}\hat{R}$$



$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\Rightarrow \nabla \cdot \vec{E} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{1}{R^2}\right) = 0$$

??

$$d\mathbf{s}_R = \hat{\mathbf{R}}R^2 \sin\theta\, d\theta\, d\phi$$
$$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\, dR\, d\phi$$
$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\, dR\, d\theta$$

$$\int_V (\nabla \cdot \vec{\mathbf{V}})\, d\tau = \underbrace{\oint_S \vec{\mathbf{V}} \cdot d\vec{\mathbf{a}}}$$

use imaginary sphere, Radius R

$$\oint \vec{E}\cdot d\vec{s} = \int \frac{1}{R^2}\hat{R} \cdot R^2 \sin\theta\, d\theta\, d\phi\, \hat{R}$$

$$= \left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

$$\int_V \nabla \cdot \vec{E}\, dV = 0 \;?$$

not a function of R

What's up??

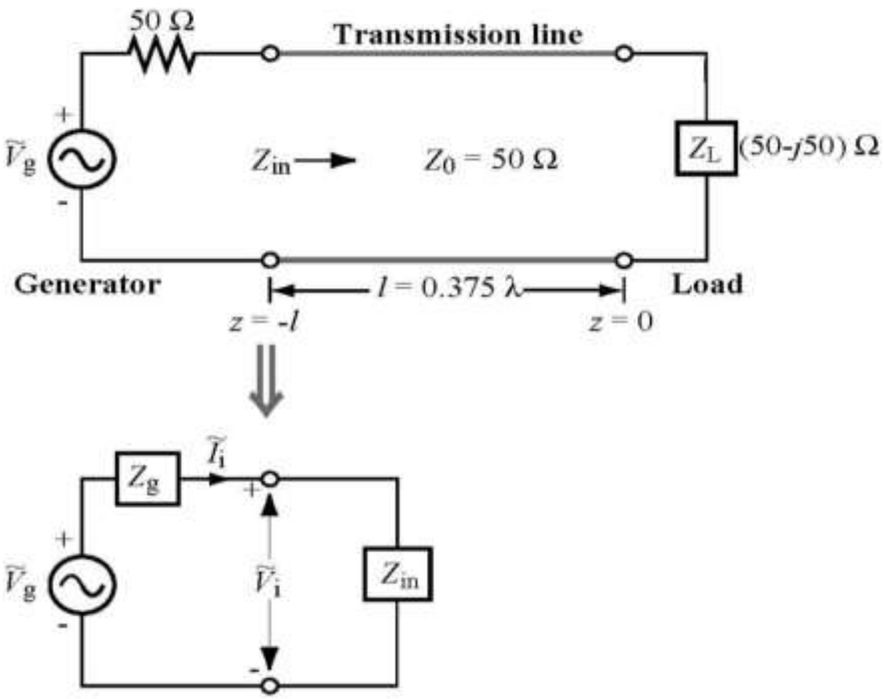
$$0 \neq 4\pi$$

Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q$

T-Line Example…

Problem 2.30 A 50-Ω lossless line of length $l = 0.375\lambda$ connects a 300-MHz generator with $\widetilde{V}_g = 300$ V and $Z_g = 50\ \Omega$ to a load Z_L. Determine the time-domain current through the load for:

(a) $Z_L = (50 - j50)\ \Omega$,
(b) $Z_L = 50\ \Omega$,
(c) $Z_L = 0$ (short circuit).



Solution:

(a) $Z_L = (50 - j50)\ \Omega$, $\beta l = \frac{2\pi}{\lambda} \times 0.375\lambda = 2.36$ (rad) $= 135°$.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j50 - 50}{50 - j50 + 50} = \frac{-j50}{100 - j50} = 0.45e^{-j63.43°}.$$

Application of Eq. (2.63) gives:

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] = 50\left[\frac{(50 - j50) + j50\tan 135°}{50 + j(50 - j50)\tan 135°}\right] = (100 + j50)\ \Omega.$$

Using Eq. (2.66) gives

$$\begin{aligned}
V_0^+ &= \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \\
&= \frac{300(100 + j50)}{50 + (100 + j50)}\left(\frac{1}{e^{j135°} + 0.45e^{-j63.43°}e^{-j135°}}\right) \\
&= 150e^{-j135°} \quad \text{(V)},
\end{aligned}$$

$$\widetilde{I}_L = \frac{V_0^+}{Z_0}(1 - \Gamma) = \frac{150e^{-j135°}}{50}(1 - 0.45e^{-j63.43°}) = 2.68e^{-j108.44°} \quad \text{(A)},$$

$$\begin{aligned}
i_L(t) &= \Re\mathfrak{e}[\widetilde{I}_L e^{j\omega t}] \\
&= \Re\mathfrak{e}[2.68e^{-j108.44°}e^{j6\pi \times 10^8 t}] \\
&= 2.68\cos(6\pi \times 10^8 t - 108.44°) \quad \text{(A)}.
\end{aligned}$$

(b)

$Z_L = 50\ \Omega$,
$\Gamma = 0$,
$Z_{in} =$ $\qquad$ $Z_0 = 50\ \Omega$,

$$V_0^+ = \frac{300 \times 50}{50 + 50}\left(\frac{1}{e^{j135°} + 0}\right) = 150e^{-j135°} \quad \text{(V)},$$

$$\widetilde{I}_L = \frac{V_0^+}{Z_0} = \frac{150}{50}e^{-j135°} = 3e^{-j135°} \quad \text{(A)},$$

$$i_L(t) = \Re\mathfrak{e}[3e^{-j135°}e^{j6\pi \times 10^8 t}] = 3\cos(6\pi \times 10^8 t - 135°) \quad \text{(A)}.$$

(c)

$Z_L = 0$,
$\Gamma = -1$,

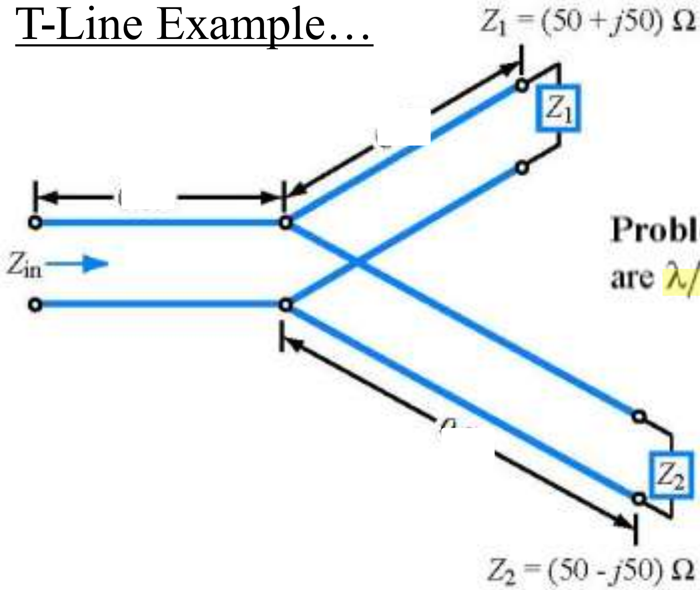
$$Z_{in} = Z_0\left(\frac{0 + jZ_0 \tan 135°}{Z_0 + 0}\right) = jZ_0 \tan 135° = -j50 \quad (\Omega),$$

$$V_0^+ = \frac{300(-j50)}{50 - j50}\left(\frac{1}{e^{j135°} - e^{-j135°}}\right) = 150e^{-j135°} \quad \text{(V)},$$

$$\widetilde{I}_L = \frac{V_0^+}{Z_0}[1 - \Gamma] = \frac{150e^{-j135°}}{50}[1 + 1] = 6e^{-j135°} \quad \text{(A)},$$

$$i_L(t) = 6\cos(6\pi \times 10^8 t - 135°) \quad \text{(A)}.$$

T-Line Example…



Problem 2.49 Repeat Problem 2.48 for the case where all three transmission lines are $\lambda/4$ in length.

Solution: Since the transmission lines are in parallel, it is advantageous to express loads in terms of admittances. In the upper branch, which is a quarter wave line,

$$Y_{1\,\text{in}} = \frac{Y_0^2}{Y_1} = \frac{Z_1}{Z_0^2},$$

and similarly for the lower branch,

$$Y_{2\,\text{in}} = \frac{Y_0^2}{Y_2} = \frac{Z_2}{Z_0^2}.$$

Thus, the total load at the junction is

$$Y_{\text{JCT}} = Y_{1\,\text{in}} + Y_{2\,\text{in}} = \frac{Z_1 + Z_2}{Z_0^2}.$$

Therefore, since the common transmission line is also quarter-wave,

$$Z_{\text{in}} = Z_0^2/Z_{\text{JCT}} = Z_0^2 Y_{\text{JCT}} = Z_1 + Z_2 = (50 + j50)\ \Omega + (50 - j50)\ \Omega = 100\ \Omega.$$

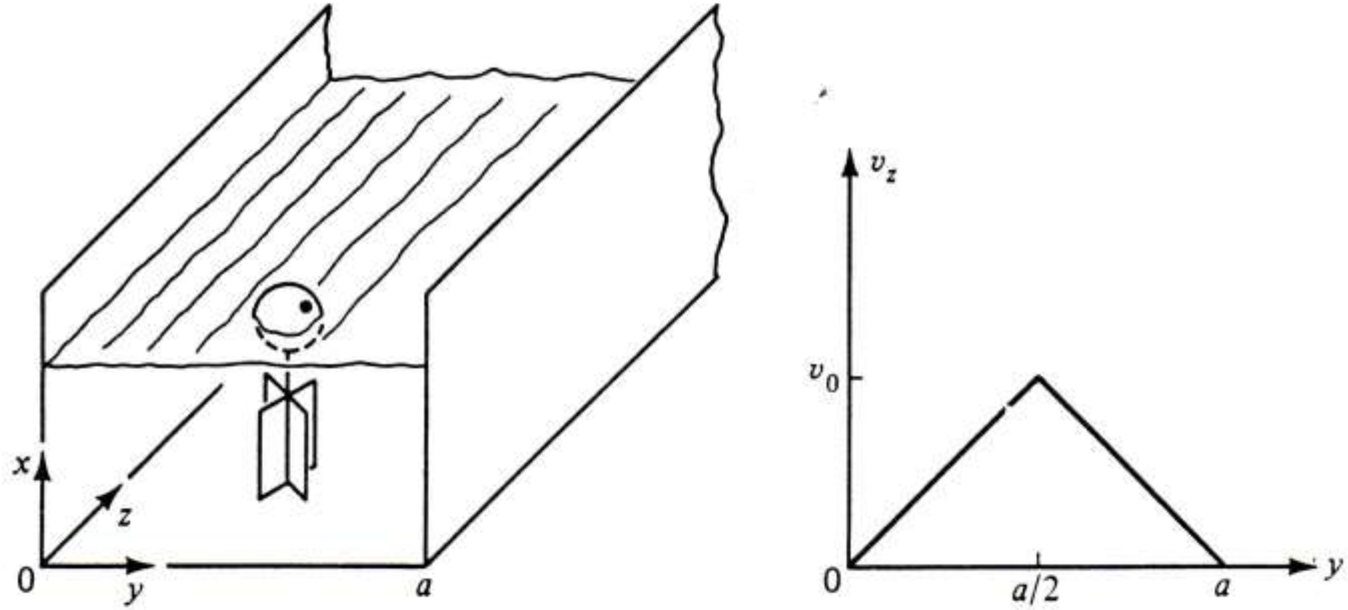
"God created the integers…
…all the rest is the work of man"

- Leopold Kronecker
http://en.wikipedia.org/wiki/Leopold_Kronecker

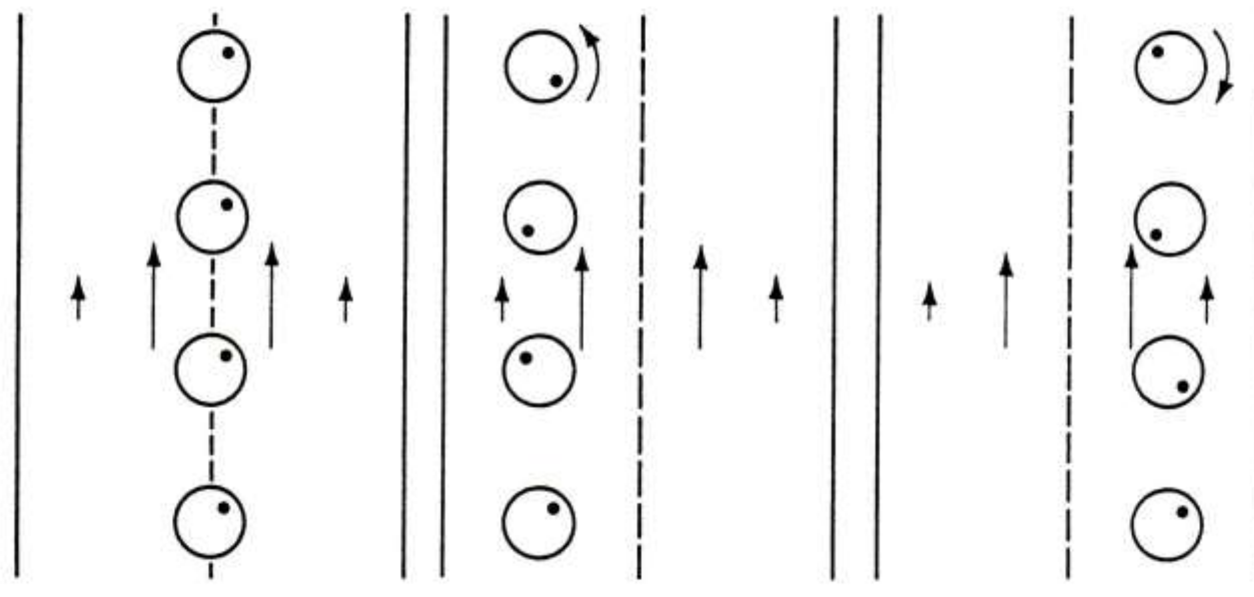
"A mathematician is a device for turning coffee into theorems."

- Paul Erdos

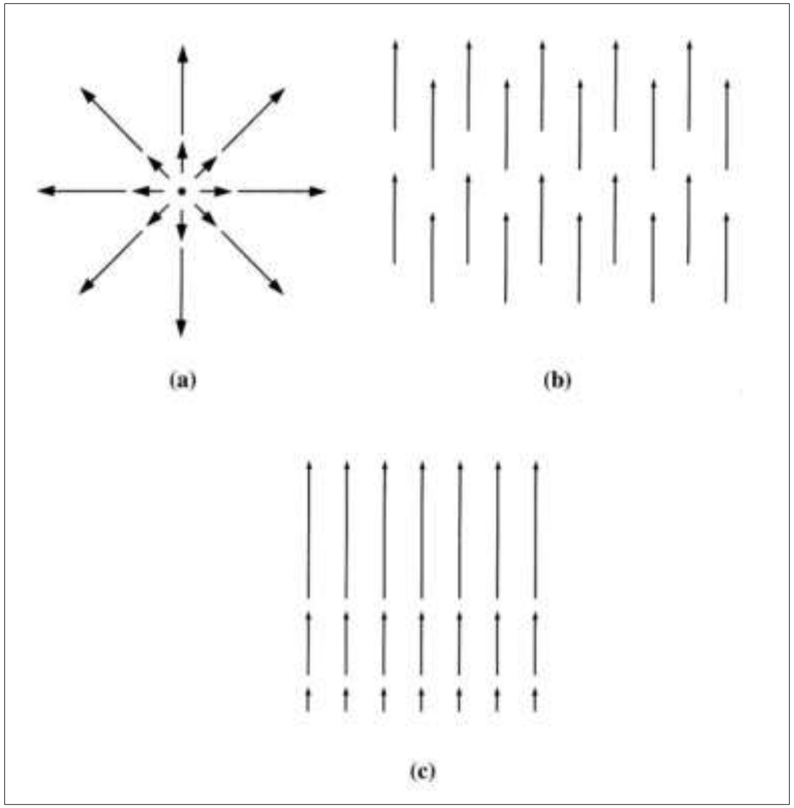
http://en.wikipedia.org/wiki/Paul_Erd%C5%91s

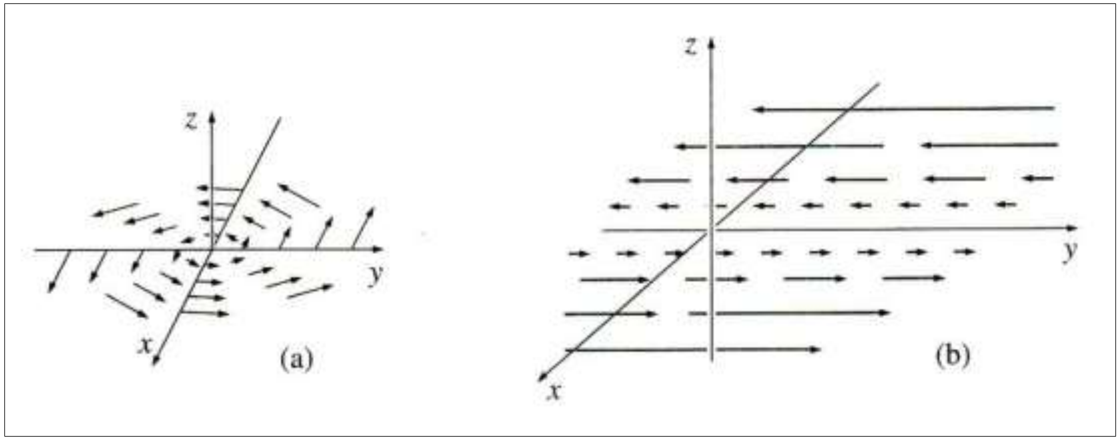


(a) (b)



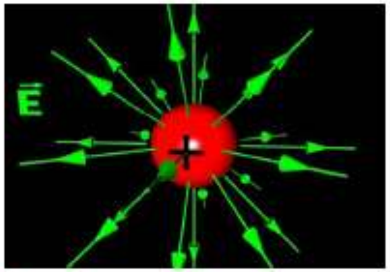
(c) (d) (e)


(a)
(b)
(c)


z
y
x
(a)
z
y
x
(b)

Vector fields have spatial variability
(EXAMPLE)

$$\vec{E} = \frac{1}{R^2}\hat{R}$$



$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

$$\Rightarrow \nabla\cdot\vec{E} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{1}{R^2}\right) = 0$$

??

$$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\, d\theta\, d\phi$$
$$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\, dR\, d\phi$$
$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\, dR\, d\theta$$

$$\int_V (\nabla\cdot\vec{\mathbf{V}})d\tau = \underbrace{\oint_S \vec{\mathbf{V}}\cdot d\vec{\mathbf{a}}}$$

use imaginary sphere, Radius R

$$\oint \vec{E}\cdot d\vec{s} = \int \frac{1}{R^2}\hat{R}\cdot R^2\sin\theta\, d\theta\, d\phi\, \hat{R}$$
$$= \left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = \underline{4\pi}$$

not a function of R

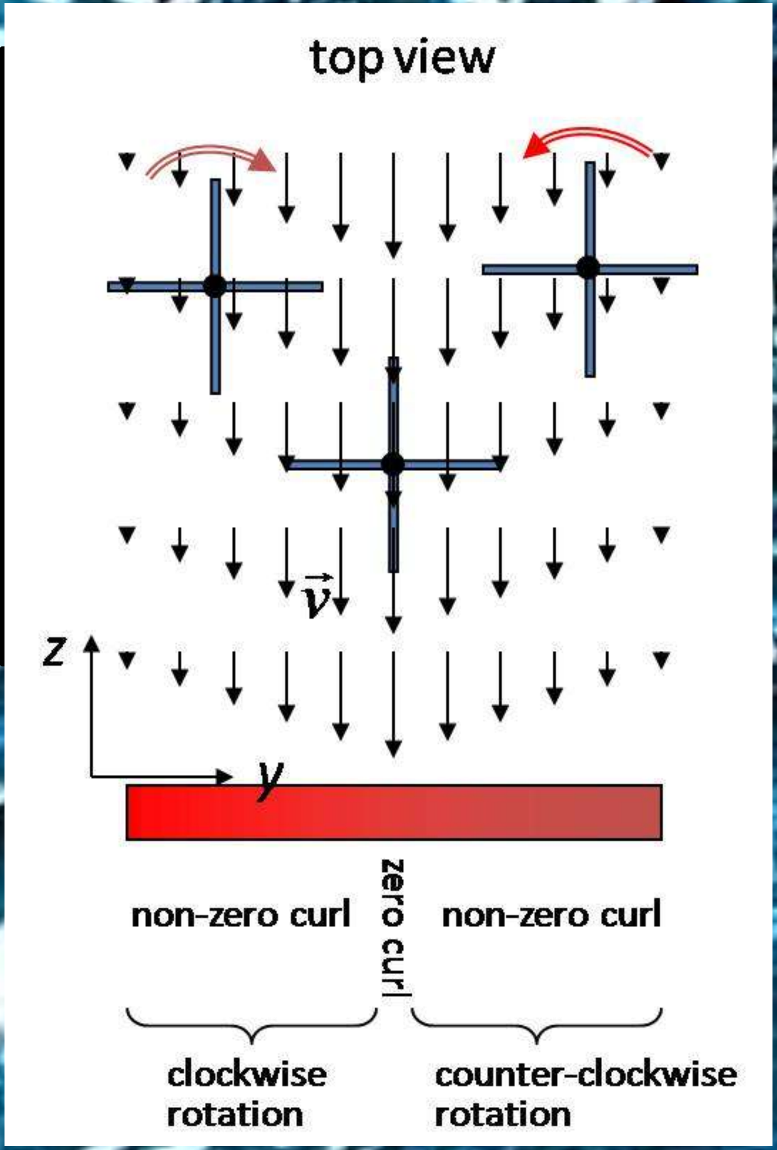
$$\int_V \nabla\cdot\vec{E}\, dV = 0\ ?$$

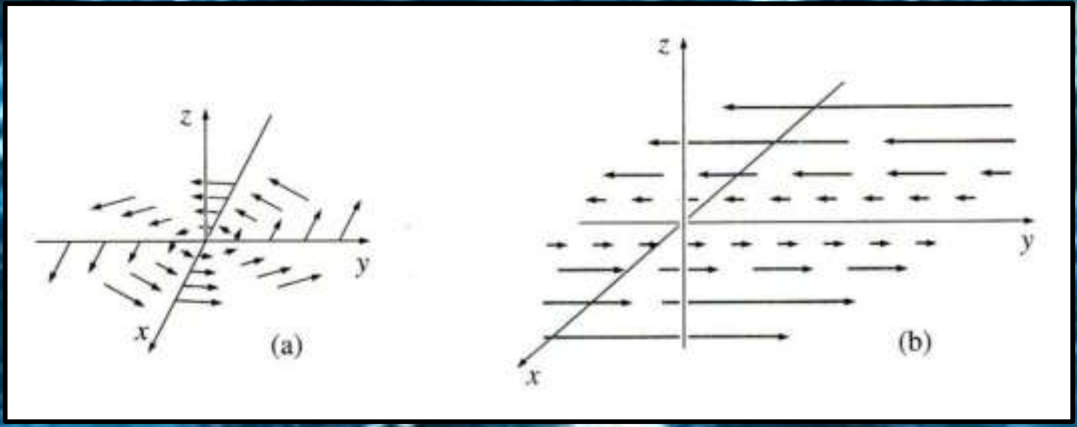
What's up??

$0 \neq 4\pi$

Gauss's law $\quad \nabla\cdot\mathbf{D} = \rho_v \quad \oint_S \mathbf{D}\cdot d\mathbf{s} = Q$

Curl


top view
$\vec{v}$
z
y
zero curl
non-zero curl
non-zero curl
clockwise rotation
counter-clockwise rotation


z
y
x
(a)
z
y
x
(b)

Vector fields have spatial variability

Say $\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$. The **mathematical** definition of Curl is given as:

$$\boxed{B_i(\bar{r}) = \lim_{\Delta s \to 0} \frac{\oint_{C_i} \mathbf{A}(\bar{r}) \cdot \overline{d\ell}}{\Delta s_i}}$$

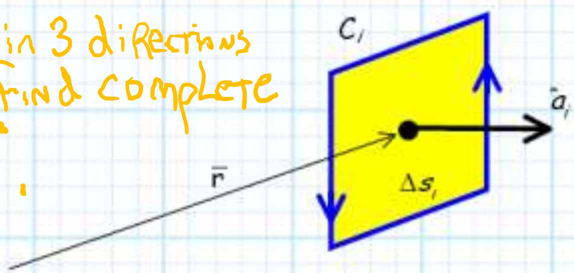
This rather complex equation requires some **explanation** !

* $B_i(\bar{r})$ is the scalar component of vector $\mathbf{B}(\bar{r})$ in the direction defined by unit vector $\hat{a}_i$ (e.g., $\hat{a}_x, \hat{a}_\rho, \hat{a}_\theta$).

* The small surface Δs_i is centered at point $\bar{r}$, and oriented such that it is normal to unit vector $\hat{a}_i$.

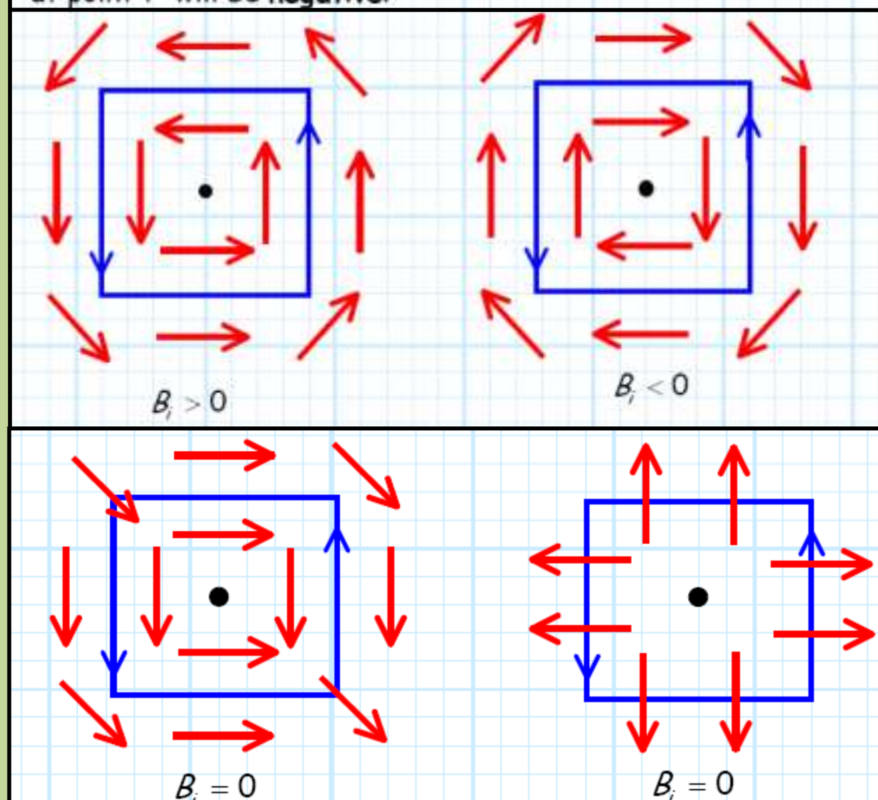
* The contour C_i is the closed contour that surrounds surface Δs_i.

Do in 3 directions to find complete $\vec{B}$.



If a component of vector field $\mathbf{A}(\bar{r})$ is pointing in the direction $\overline{d\ell}$ at every point on contour C_i (i.e., **tangential** to the contour). Then the line integral, and thus the curl, will be **positive**.

If, however, a component of vector field $\mathbf{A}(\bar{r})$ points in the opposite direction ($-\overline{d\ell}$) at every point on the contour, the curl at point $\bar{r}$ will be **negative**.



Note: these are vanishingly *small* contours…

Vector fields have spatial variability

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Example

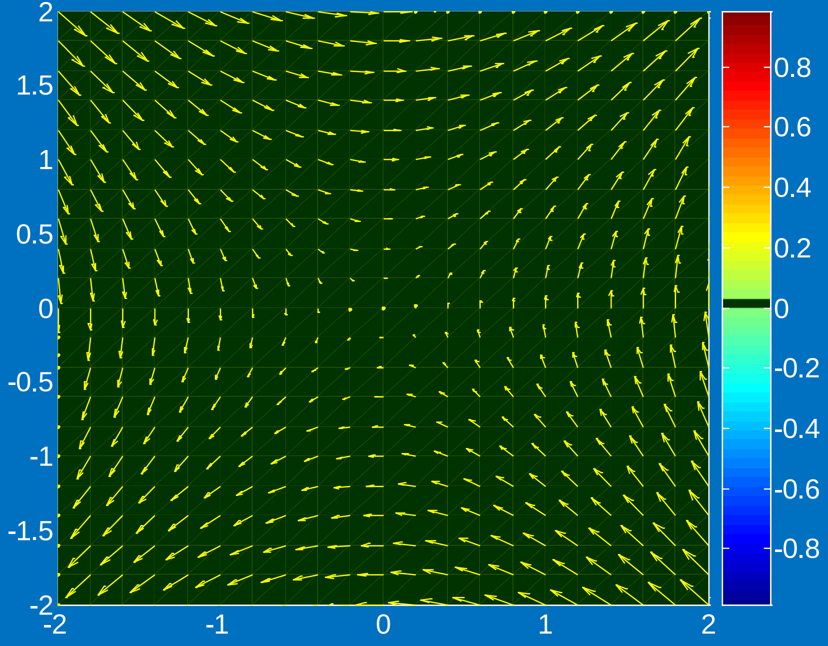
$$\vec{A} = \hat{x}(1+2yz) - \hat{y}\left(x^2+y^2\right) + \hat{z}(1-2yz) \qquad \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1+2yz & -\left(x^2+y^2\right) & 1-2yz \end{vmatrix} = \hat{x}(-2z) + \hat{y}2y + \hat{z}(-2x-2z)$$

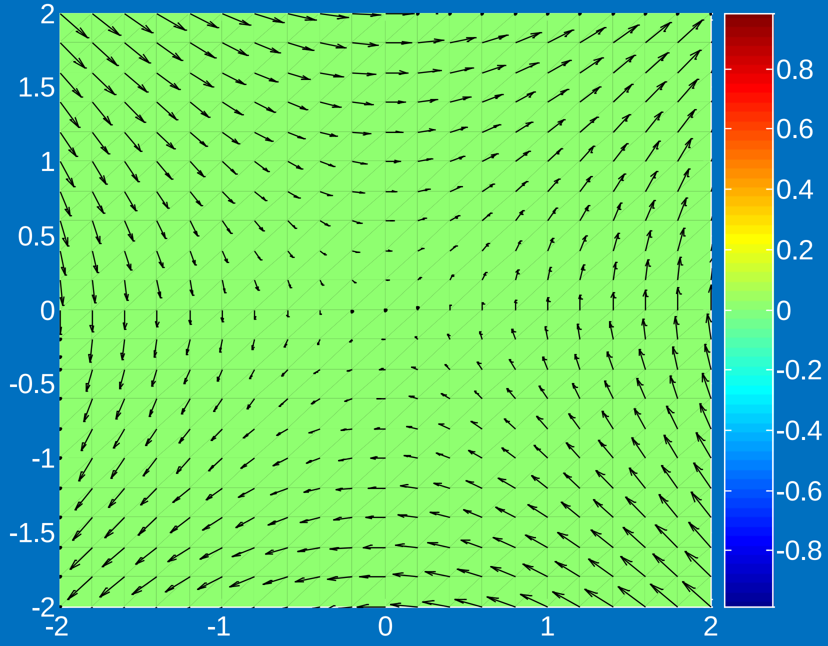
$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} r & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} R & \hat{\boldsymbol{\phi}} R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

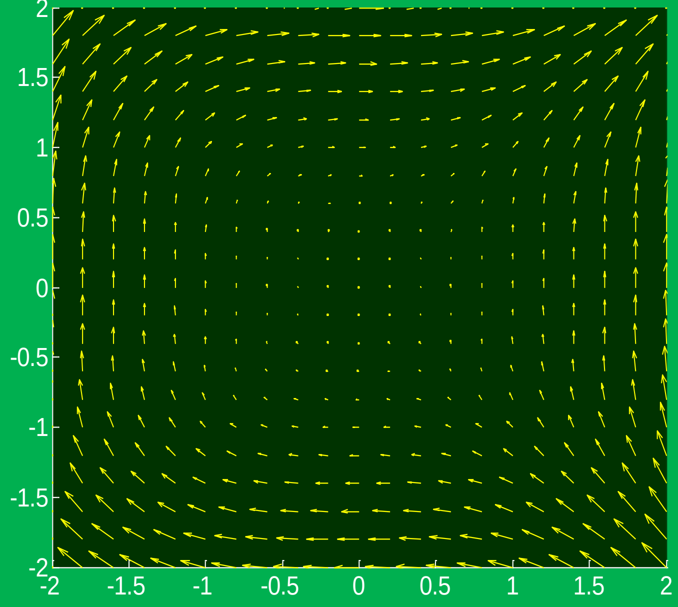
$$\vec{\mathbf{V}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \qquad \nabla \times \vec{\mathbf{V}} = 1 - 1 = 0$$

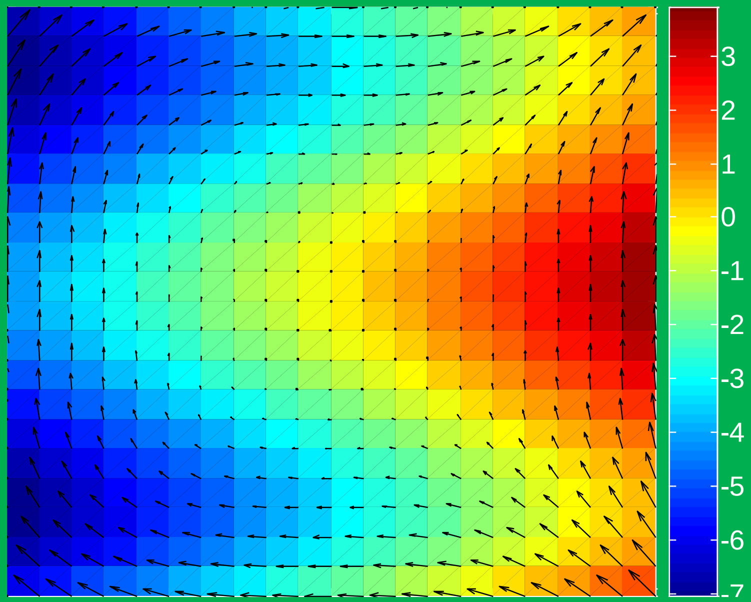




$$\vec{\mathbf{V}} = y^2 \sin(y)\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

$$\nabla \times \vec{\mathbf{V}} = \left(2x - 2y\sin(y) + y^2\cos(y)\right)\hat{\mathbf{z}}$$



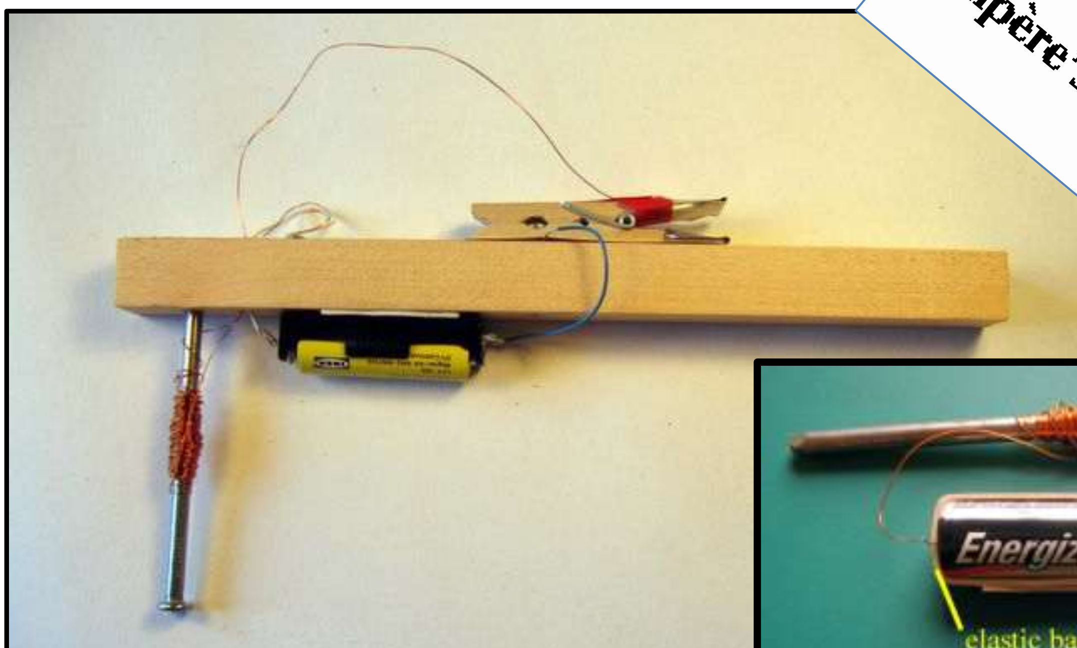


<u>Why DO we care about curl</u> ???

<u>Helmholtz Theorem</u>

*If you know $\nabla \cdot \mathbf{A}(\bar{r})$ **and** you know $\nabla \times \mathbf{A}(\bar{r})$, **you** have enough information to determine the vector field $\mathbf{A}(\bar{r})$!*

Is this true????



Ampère's law

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

Stokes' Theorem

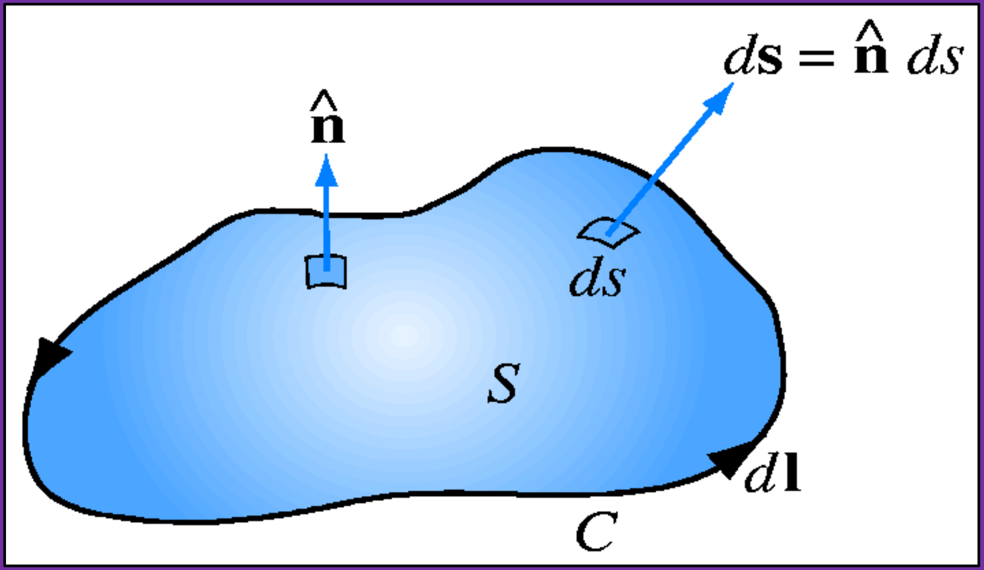
http://en.wikipedia.org/wiki/George_Gabriel_Stokes

$$\int_S (\nabla \times \vec{\mathbf{v}}) \cdot d\vec{\mathbf{a}} = \oint_P \vec{\mathbf{v}} \cdot d\vec{\mathbf{l}}$$

surface

OR $d\vec{s}$ (normal to surface)

Path (OR Contour) around edge of surface



http://www.math.umn.edu/~rogness/multivar/multiple_surfaces.shtml

BTW… variables are, well, variable !! ☺

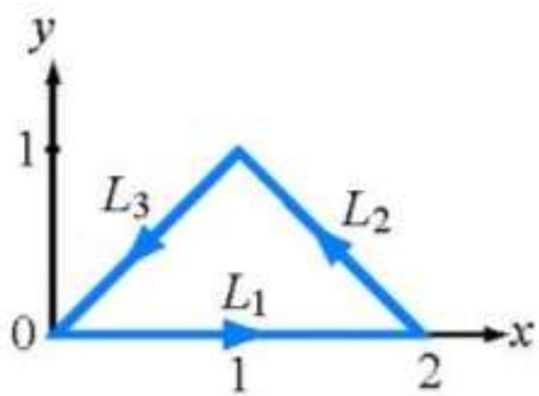
Q: what if *S* is a closed surface??

For the vector field $\mathbf{E} = \hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)$, calculate

(a) $\oint_C \mathbf{E} \cdot d\mathbf{l}$ around the triangular contour shown

(b) $\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s}$ over the area of the triangle.

Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$



$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $y = 2 - x$, and $y = x$, respectively.

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$L_1 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=0}^{2} (xy)|_{y=0,z=0}\,dx - \int_{y=0}^{0} (x^2 + 2y^2)|_{z=0}\,dy + \int_{z=0}^{0} (0)|_{y=0}\,dz = 0,$$

$$L_2 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=2}^{1} (xy)|_{z=0,y=2-x}\,dx - \int_{y=0}^{1} (x^2 + 2y^2)|_{x=2-y,z=0}\,dy + \int_{z=0}^{0} (0)|_{y=2-x}\,dz$$
$$= \left(x^2 - \frac{x^3}{3}\right)\Big|_{x=2}^{1} - (4y - 2y^2 + y^3)\Big|_{y=0}^{1} + 0 = \frac{-11}{3},$$

$$L_3 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{0} (xy)|_{y=x,z=0}\,dx - \int_{y=1}^{0} (x^2 + 2y^2)|_{x=y,z=0}\,dy + \int_{z=0}^{0} (0)|_{y=x}\,dz$$
$$= \left(\frac{x^3}{3}\right)\Big|_{x=1}^{0} - (y^3)\Big|_{y=1}^{0} + 0 = \frac{2}{3}.$$

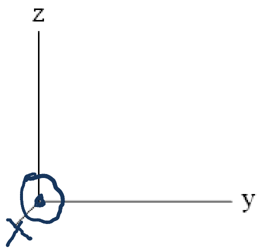
Therefore,

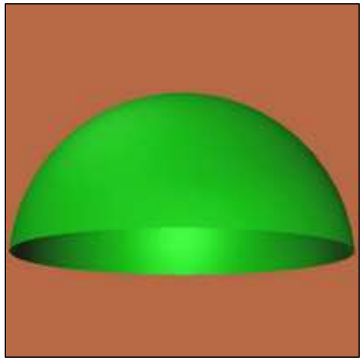
$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) From Eq. (3.105), $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\iint \nabla \times \mathbf{E} \cdot d\mathbf{s} = \int_{x=0}^{1} \int_{y=0}^{x} ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx))|_{z=0}$$
$$+ \int_{x=1}^{2} \int_{y=0}^{2-x} ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx))|_{z=0}$$
$$= -\int_{x=0}^{1} \int_{y=0}^{x} 3x\,dy\,dx - \int_{x=1}^{2} \int_{y=0}^{2-x} 3x\,dy\,dx$$
$$= -\int_{x=0}^{1} 3x(x - 0)\,dx - \int_{x=1}^{2} 3x((2 - x) - 0)\,dx$$
$$= -(x^3)\Big|_0^1 - (3x^2 - x^3)\Big|_{x=1}^{2} = -3.$$

Problem 3.47 Verify Stokes's Theorem for the vector field $\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta$ by evaluating it on the hemisphere of unit radius.





Solution:

$$\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi.$$

Hence, $A_R = \cos\theta$, $A_\theta = 0$, $A_\phi = \sin\theta$.

$$\nabla\times\mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$
$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial\theta}\right]$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\right) - \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \hat{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial A_R}{\partial\theta}$$
$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) - \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial R}(R\sin\theta) - \hat{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial}{\partial\theta}(\cos\theta)$$
$$= \hat{\mathbf{R}}\frac{2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\frac{\sin\theta}{R}.$$

For the hemispherical surface, $d\mathbf{s} = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$.

Differential surface areas	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}(\nabla\times\mathbf{A})\cdot d\mathbf{s}$$
$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\frac{\hat{\mathbf{R}}2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\frac{\sin\theta}{R}\right)\cdot\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\bigg|_{R=1}$$
$$= 4\pi R\left.\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right|_{R=1} = 2\pi.$$

The contour C is the circle in the x–y plane bounding the hemispherical surface.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi}(\hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta)\cdot\hat{\boldsymbol{\phi}}R\,d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = R\sin\theta\int_0^{2\pi}d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = 2\pi.$$

Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$

Laplacian

The scalar Laplacian is simply the **divergence** of the **gradient** of a scalar field:

$$\nabla \cdot \nabla g(\bar{r})$$

The scalar Laplacian therefore both **operates** on a scalar field and **results** in a scalar field.

Often, the Laplacian is denoted as "∇^2", i.e.:

$$\nabla^2 g(\bar{r}) \doteq \nabla \cdot \nabla g(\bar{r})$$

From the expressions of divergence and gradient, we find that the scalar Laplacian is expressed in **Cartesian** coordinates as:

$$\boxed{\nabla^2 g(\bar{r}) = \frac{\partial^2 g(\bar{r})}{\partial x^2} + \frac{\partial^2 g(\bar{r})}{\partial y^2} + \frac{\partial^2 g(\bar{r})}{\partial z^2}}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

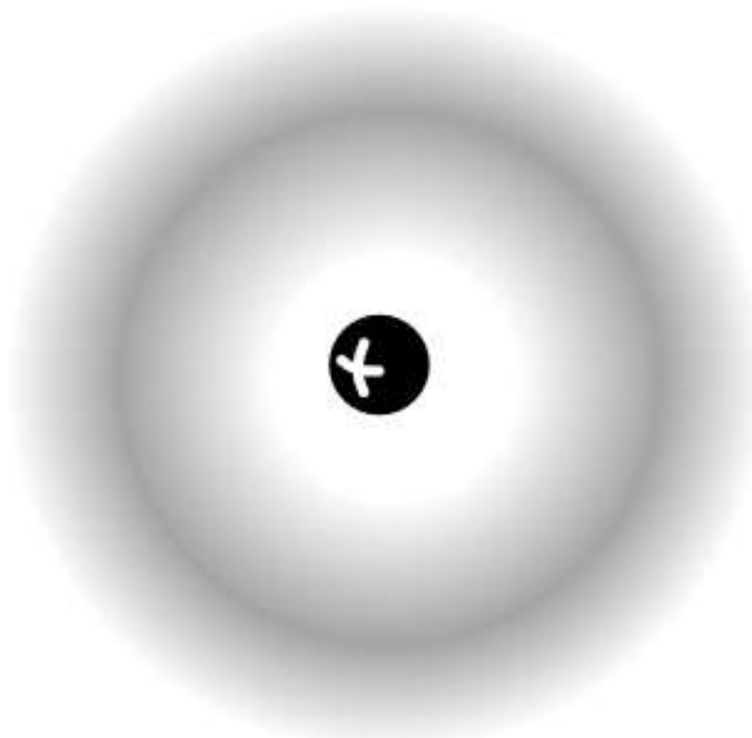
Example:

Find $\nabla^2 \vec{A}$, where $\vec{A} = \hat{x}(1+2yz) - \hat{y}\left(x^2 + y^2\right) + \hat{z}(1-2yz)$

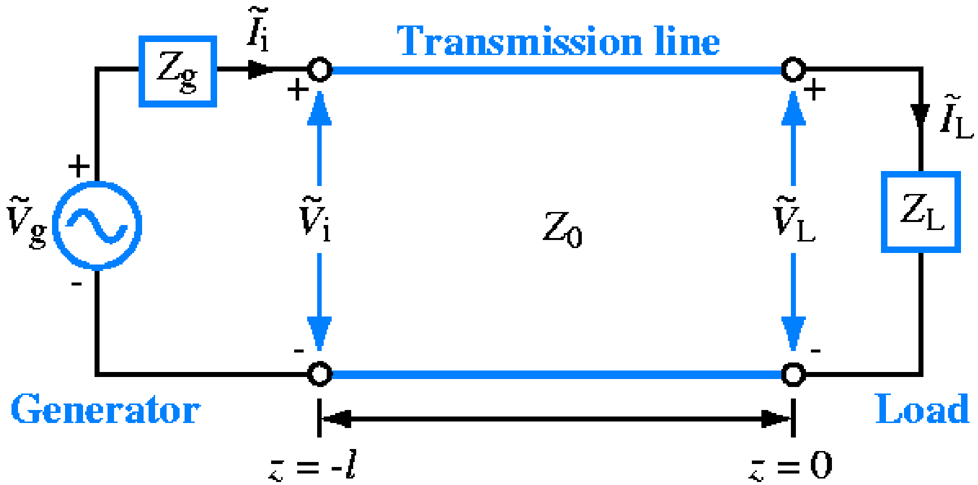
$$\nabla^2 \vec{A} =$$

$$= \hat{x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(1+2yz) + \hat{y}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(-x^2 - y^2\right) + \hat{z}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(1-2yz) =$$

$$= \hat{x}0 + \hat{y}(-2-2) + \hat{z}0 = -\hat{y}4$$



Why DO we care about the Laplacian ???



$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z)$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z)$$

$$\frac{d^2\tilde{V}(z)}{dz^2} = \gamma^2\,\tilde{V}(z)$$

∇^2 is the analog of this in 3-D for vector fields (Like $\vec{E}$)

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$\tilde{V}(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} \Rightarrow v(z,t) = V_0^+ e^{-\alpha z}\cos(\omega t - \beta z)$$

Traveling Wave ☺

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

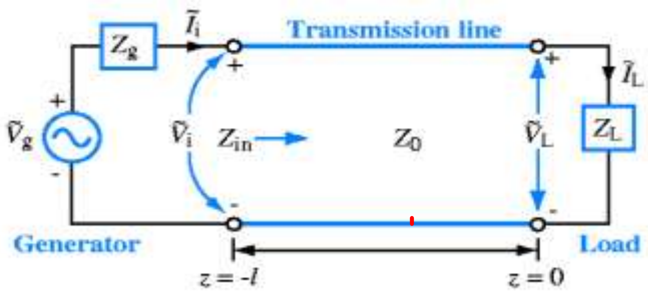
(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

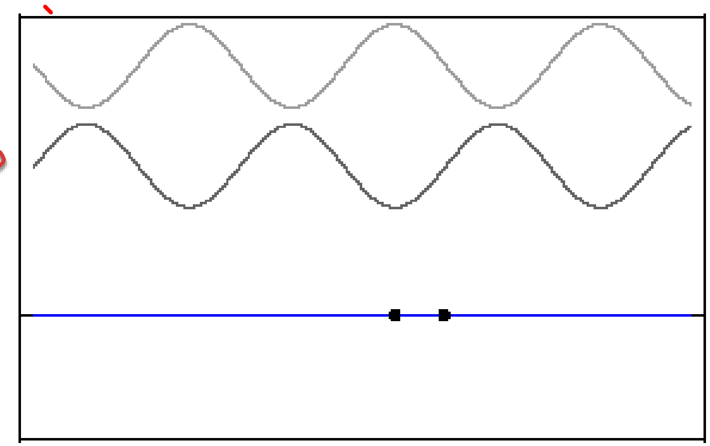
(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$



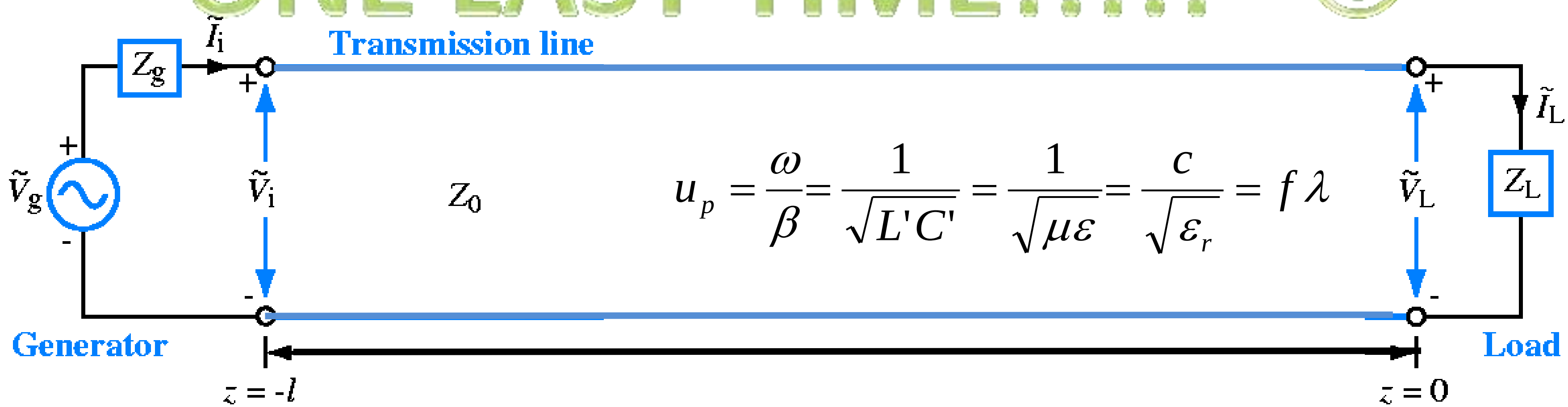
$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$v(z,t) = \left|V_0^+\right| e^{-\alpha z} \cos\left(\omega t - \beta z + \phi^+\right) + \left|V_0^-\right| e^{\alpha z} \cos\left(\omega t + \beta z + \phi^-\right)$$

phase



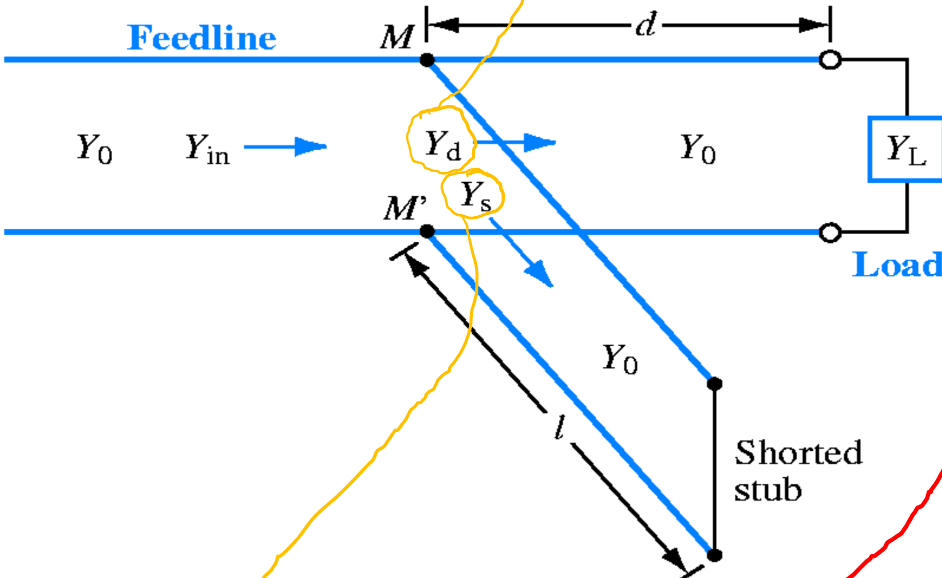
ONE LAST TIME!!!!!



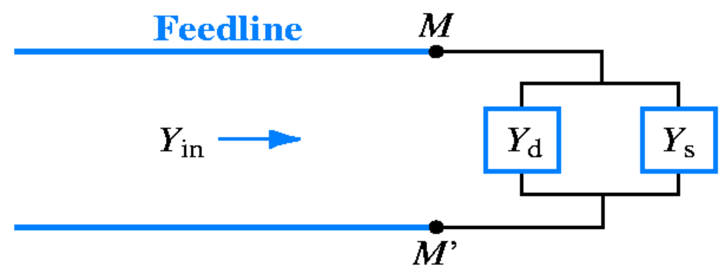
$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = f\lambda$$

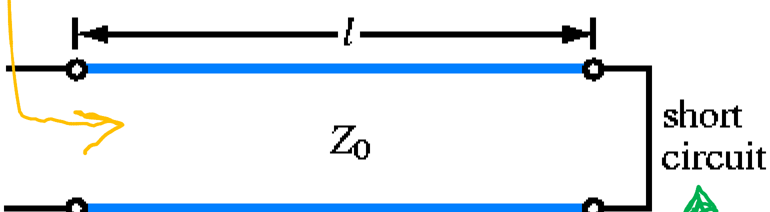
Sinusoidal Steady State vs. Transients

Step-by-step stub (cont.)



Y_d is input admittance to Y_L plus T-line of length d (it does not include the stub or feedline)





How's this all change if this is an open circuit

Another quick T-line thought:

at 60 GHz, wavelength is less than 5 mm…

…how does this impact circuit design at this frequency?

15) (20 pts.) Given vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\boldsymbol{A} = e^{-(x+y+z)}[\hat{x} + \hat{y} + \hat{z}] \; and \;\; \boldsymbol{B} = 10\hat{x} + 10\hat{y}$$

find:

$\boldsymbol{A}\cdot\boldsymbol{B} =$ ______________________

$\boldsymbol{A}\times\boldsymbol{B} =$ ______________________

$\nabla\cdot\boldsymbol{A} =$ ______________________

$\nabla\times\boldsymbol{A} =$ ______________________

Finally, what is the angle between the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ at the point P(3,0,π)?

____

10) Vectors in spherical coordinates can **not** be added directly. T F

11) Voltage waves always travel at c down a lossless transmission line. T F

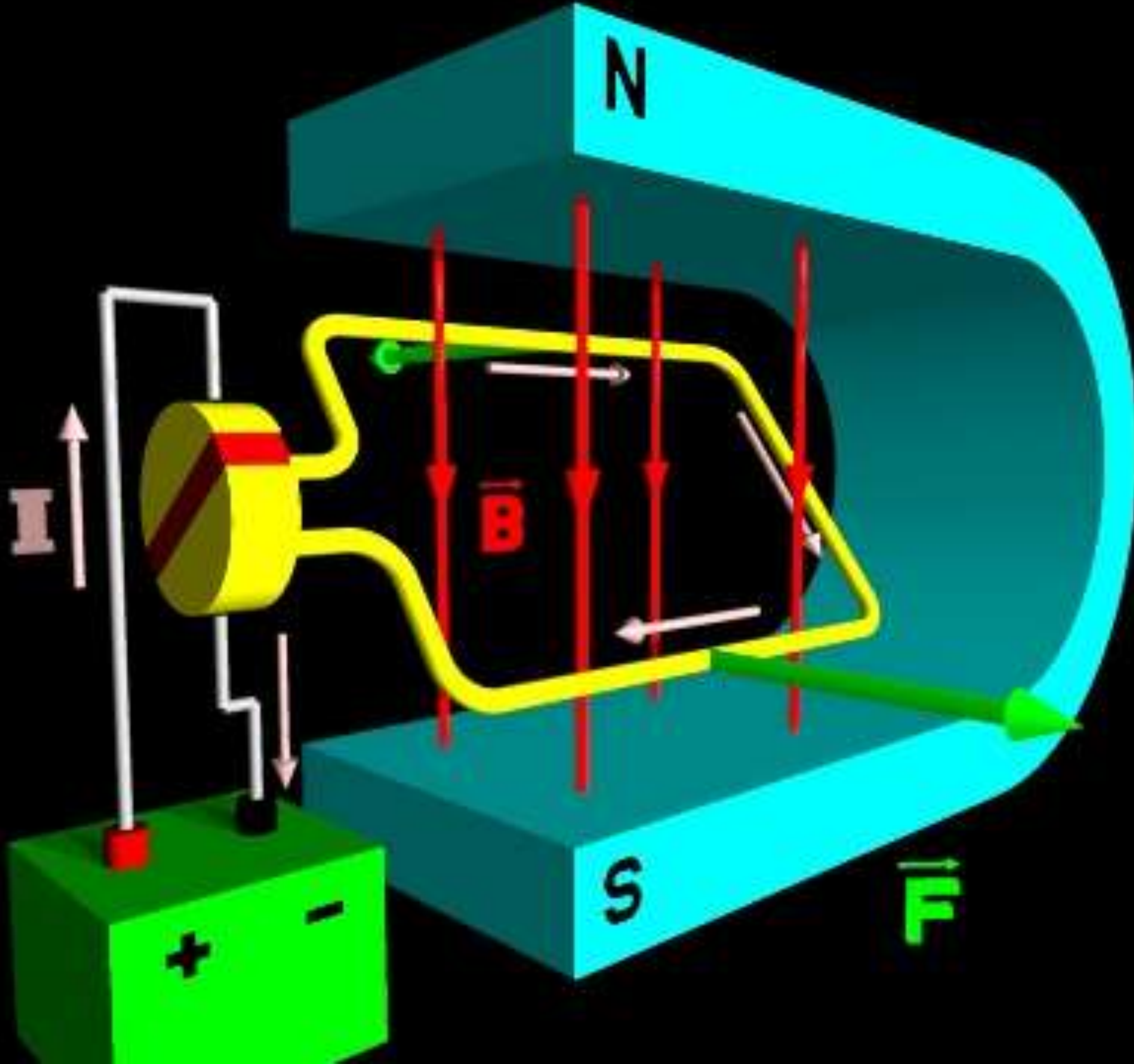
If I know the gradient and the divergence at all points (as well as some boundary conditions) I can reproduce the field exactly. T F

j^j has infinitely many values, all of which are real numbers. T F

The input impedance of a shorted stub can take on any value, depending on its length. T F

A given shorted stub behaves like an open circuit at 600 MHz. What is the closest frequency at which it behaves like a short (assuming constant phase speed)?

If $\boldsymbol{A} = A_r \cos\theta\, \widehat{\boldsymbol{R}}$, evaluate $\oiint \nabla \times \boldsymbol{A} \cdot d\boldsymbol{s}$, where the integration is over the surface of the unit sphere.



N
I
$\vec{B}$
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+
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